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2	0.3010300	19	1.2787536	43	1.6334685	71	1.8512583
3	0.4771213	23	1.3617278	47	1.6720979	73	1.8633229
7	0.8450980	29	1.4623980	53	1.7242759	79	1.8976271
11	1.0413927	31	1.4913617	59	1.7708520	83	1.9190781
13	1.1139434	37	1.5682017	61	1.7853298	89	1.9493900
17	1.2304489	41	1.6127839	67	1.8260748	97	1.9867717

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PLANE TRIGONOMETRY.

PART II.

CHAPTER I.

ON THE SERIES REQUIRED FOR THE CALCULATION OF π

THE special object of this Chapter is to shew the different steps, by which we arrive at the value of the number π, which is of such great importance in Trigonometry, and indeed in every branch of Mathematics. But, in the course of our reasoning, many interesting results and remarkable formulæ will arise.

1. *To shew that*

$$(\cos\alpha+\sqrt{-1}\sin\alpha)(\cos\beta+\sqrt{-1}\sin\beta)\ldots(\cos\lambda+\sqrt{-1}\sin\lambda)$$
$$=\cos(\alpha+\beta+\ldots+\lambda)+\sqrt{-1}\sin(\alpha+\beta+\ldots+\lambda).$$

We have by Mult^{n} $(\cos\alpha+\sqrt{-1}\sin\alpha)(\cos\beta+\sqrt{-1}\sin\beta)$
$=(\cos\alpha\cos\beta-\sin\alpha\sin\beta)+\sqrt{-1}(\sin\alpha\cos\beta+\cos\alpha\sin\beta)$
$=(\cos(\alpha+\beta)+\sqrt{-1}\sin(\alpha+\beta)$.

Suppose this Law true for any number of such factors, so that
$(\cos\alpha+\sqrt{-1}\sin\alpha)(\cos\beta+\sqrt{-1}\sin\beta)\ldots(\cos\kappa+\sqrt{-1}\sin\kappa)$
$=\cos(\alpha+\beta+\ldots\kappa)+\sqrt{-1}\sin(\alpha+\beta+\ldots\kappa)$
$=\cos\varsigma+\sqrt{-1}\sin\varsigma$, suppose;
then, introducing another factor, $\cos\lambda+\sqrt{-1}\sin\lambda$, we get
$(\cos\alpha+\sqrt{-1}\sin\alpha)\ldots(\cos\kappa+\sqrt{-1}\sin\kappa)(\cos\lambda+\sqrt{-1}\sin\lambda)$
$=(\cos\varsigma+\sqrt{-1}\sin\varsigma)(\cos\lambda+\sqrt{-1}\sin\lambda)$
$=\cos(\varsigma+\lambda)+\sqrt{-1}\sin(\varsigma+\lambda)$ by the previous result,
$=\cos(\alpha+\beta+\ldots+\kappa+\lambda)+\sqrt{-1}\sin(\alpha+\beta+\ldots+\kappa+\lambda)$.

Hence, since the Law has been shewn to be true for *two* factors, it is also true for *three*, and, therefore, for *four*, and so on, by Induction, for any number of such factors.

2. *To express* $\tan(\alpha+\beta+\gamma+\&c.)$ *in terms of* $\tan\alpha$, $\tan\beta$, $\tan\gamma$, &c.

From the result of [1] we obtain

$$\cos(\alpha+\beta+\gamma+\&c)+\sqrt{-1}\sin(\alpha+\beta+\gamma+\&c)$$
$$=(\cos\alpha+\sqrt{-1}\sin\alpha)(\cos\beta+\sqrt{-1}\sin\beta)(\cos\gamma+\sqrt{-1}\sin\gamma)\ldots$$
$$=\cos\alpha(1+\sqrt{-1}\tan\alpha)\cos\beta(1+\sqrt{-1}\tan\beta)\cos\gamma(1+\sqrt{-1}\tan\gamma)\ldots$$
$$=\cos\alpha\cos\beta\cos\gamma\ldots(1+\sqrt{-1}\tan\alpha)(1+\sqrt{-1}\tan\beta)(+\sqrt{-1}\tan\gamma)\ldots$$
$$=\cos\alpha\cos\beta\cos\gamma\ldots\{1+\sqrt{-1}S_1-S_2-\sqrt{-1}S_3+S_4+\&c\},\text{ by Mult}^{n},$$

where $S_1=\tan\alpha+\tan\beta+\tan\gamma+\ldots$

$S_2=\tan\alpha\tan\beta+\tan\alpha\tan\gamma+\tan\beta\tan\gamma+\ldots$

$S_3=\tan\alpha\tan\beta\tan\gamma+\&c$, &c=&c.

Therefore, equating possible and impossible parts of this expression, we have

$$\cos(\alpha+\beta+\gamma+\&c)=\cos\alpha\cos\beta\cos\gamma\ldots(1-S_2+S_4-\&c),$$
$$\sin(\alpha+\beta+\gamma+\&c)=\cos\alpha\cos\beta\cos\gamma\ldots(S_1-S_3+S_5-\&c),$$

and, therefore, $\tan(\alpha+\beta+\gamma+\&c)=\dfrac{S_1-S_3+S_5-\&c}{1-S_2+S_4-\&c}$.

Let the number of angles be n: then, if n be odd, the last term of the num^{r} will be $(-1)^{\frac{1}{2}(n-1)}S_n$, and of the den^{r} $(-1)^{\frac{1}{2}(n-1)}S_{n-1}$; or, if n be even, they will be $(-1)^{\frac{1}{2}(n-2)}S_{n-1}$, $(-1)^{\frac{1}{2}n}S_n$, respectively.

If in the above we suppose α, β, &c, each equal to θ, we may get from the preceding results the values of $\sin n\theta$, $\cos n\theta$, $\tan n\theta$, when n is a positive integer, in terms of θ. But this may be better obtained by proving first the following Theorem.

3. DEMOIVRE'S THEOREM. *To shew that for all values of* m, $(\cos\theta\pm\sqrt{-1}\sin\theta)^m=\cos m\theta\pm\sqrt{-1}\sin m\theta$.

We need not trouble ourselves with the double sign; for, if we can shew that $(\cos\theta+\sqrt{-1}\sin\theta)^m=\cos m\theta+\sqrt{-1}\sin m\theta$, then, writing $-\theta$ for θ, we get at once

$$(\cos\theta-\sqrt{-1}\sin\theta)^m=\cos m\theta-\sqrt{-1}\sin m\theta.$$

First, let the index be a *positive integer*.

In the result of [1] let there be m such factors, each of them being $\cos\theta+\sqrt{-1}\sin\theta$:

then $(\cos\theta+\sqrt{-1}\sin\theta)^m=\cos(\theta+\theta+\&c)\times\sqrt{-1}\sin(\theta+\theta+\&c)$
$$=\cos m\theta+\sqrt{-1}\sin m\theta.$$

Next, let the index be a *negative integer*.

Then $(\cos\theta+\sqrt{-1}\sin\theta)^{-m}$

$=\dfrac{1}{(\cos\theta+\sqrt{-1}\sin\theta)^m}=\dfrac{1}{\cos m\vartheta+\sqrt{-1}\sin m\vartheta}$, by the former case,

$=\dfrac{\cos^2 m\theta+\sin^2 m\theta}{\cos m\vartheta+\sqrt{-1}\sin m\theta}=\cos m\theta-\sqrt{-1}\sin m\vartheta$, by common Div^{n},

$=\cos(-m\theta)+\sqrt{-1}\sin(-m\theta)$.

Lastly, let the index be a *fraction*, positive or negative.

Then, since

$$\left(\cos\frac{m}{n}\theta+\sqrt{-1}\sin\frac{m}{n}\theta\right)^n=\cos m\theta+\sqrt{-1}\sin m\theta=(\cos\theta+\sqrt{-1}\sin\theta)^m,$$

therefore, taking the n^{th} root on both sides, we have

$$\{\cos\theta+\sqrt{-1}\sin\theta\}^{\frac{m}{n}}=\cos\frac{m}{n}\theta+\sqrt{-1}\sin\frac{m}{n}\theta.$$

In this last case, however, it should be noticed that the quantity $(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{m}{n}}$, in which an n^{th} root is to be taken, has (by the principles of Algebra) n distinct values, and that $\cos\frac{m}{n}\theta+\sqrt{-1}\sin\frac{m}{n}\theta$ is only one of these.

Hence then, for all values of m, positive or negative, integral or fractional, we have, as above stated,

$$(\cos\theta\pm\sqrt{-1}\sin\theta)^m=\cos m\theta\pm\sqrt{-1}\sin m\theta.$$

4. We may thus find the n values, above mentioned, of

$$(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{m}{n}}.$$

Since $\cos\theta=\cos(2r\pi+\theta)$, and $\sin\theta=\sin(2r\pi+\theta)$,

$$\therefore(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{m}{n}}=\{\cos(2r\pi+\theta)+\sqrt{-1}\sin(2r\pi+\theta)\}^{\frac{m}{n}}$$

$$=\cos\frac{m}{n}(2r\pi+\theta)+\sqrt{-1}\sin\frac{m}{n}(2r\pi+\theta);$$

which last expression, by writing 0, 1, 2, &c. $(n-1)$, for r, will have n different values, and no more.

For (i), if we write 0, 1, 2, &c. $(n-1)$, for r, we get n different values for it; since, if any two were identical, as, for instance, when $r=p$ and when $r=q$, (where p and q are each less than n,) then the angles $\frac{m}{n}(2p\pi+\theta)$ and $\frac{m}{n}(2q\pi+\theta)$ must differ by a multiple of 2π, or $\frac{m}{n}(p-q)$ must be an integer. But this is impossible, since m is prime to n, and p, q, are each less than n.

And (ii), if we take for r any value beyond the limits 0 and $(n-1)$, we shall get no new value; as, for instance, if $r=kn+r'$, (where r' is positive and less than n, and k may be any integer, positive or negative,) then

$$\cos\frac{m}{n}\{2(kn+r')\pi+\theta\}+\sqrt{-1}\sin\frac{m}{n}\{2(kn+r')\pi+\theta\}$$

$$=\cos\{2km\pi+\frac{m}{n}(2r'\pi+\theta)\}+\sqrt{-1}\sin\{2km\pi+\frac{m}{n}(2r'\pi+\theta)\}$$

$$=\cos\frac{m}{n}(2r'\pi+\theta)+\sqrt{-1}\sin\frac{m}{n}(2r'\pi+\theta),$$

which is one of the values before obtained.

Cor. 1. If $m=1$, then

$$(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{1}{n}}=\cos\frac{2r\pi+\theta}{n}+\sqrt{-1}\sin\frac{2r\pi+\theta}{n}.$$

Cor. 2. Since $(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{m}{n}}$ may be regarded as expressing, (i) $\{(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{1}{n}}\}^m$, or (ii) $\{(\cos\theta+\sqrt{-1}\sin\theta)^m\}^{\frac{1}{n}}$, that is,

(i) $$\left(\cos\frac{2r\pi+\theta}{n}+\sqrt{-1}\sin\frac{2r\pi+\theta}{n}\right)^m=\cos\frac{m}{n}(2r\pi+\theta)+\sqrt{-1}\sin\frac{m}{n}(2r\pi+\theta),$$

or (ii) $$(\cos m\theta+\sqrt{-1}\sin m\theta)^{\frac{1}{n}}=\cos\frac{2r\pi+m\theta}{n}+\sqrt{-1}\sin\frac{2r\pi+m\theta}{n},$$

the two values thus resulting must be identical. And this is easily shown to be the case: for, taking r from 1 to $(n-1)$ in the former, the multiples of 2π will be $\frac{m}{n}$, $\frac{2m}{n}$, $\frac{3m}{n}$, ... $\frac{(n-1)m}{n}$; and, if the division by n be performed, no two of these fractions can give the same remainder; since, if $\frac{pm}{n}$ and $\frac{qm}{n}$ were to do this, then their difference $\frac{(p-q)m}{n}$ would be an integer, which is impossible, since p, q, are each less than n, and m prime to it. Hence the $(n-1)$ remainders, being all different and each less than n, will include all the terms of the series 1, 2, 3, ... $(n-1)$ and, consequently, suppressing the multiples of 2π, the n values of the former expression will be identical with those of the latter.

5. *To express* $\sin n\theta$ *and* $\cos n\theta$ *in terms of* θ.

By Demoivre's Theorem, we have

$$\cos n\theta+\sqrt{-1}\sin n\theta=(\cos\theta+\sqrt{-1}\sin\theta)^n=\cos^n\theta(1+\sqrt{-1}\tan\theta)^n$$

$$=\cos^n\theta(1+n\sqrt{-1}\tan\theta-\frac{n(n-1)}{1.2}\tan^2\theta-\frac{n(n-1)(n-2)}{1.2.3}\sqrt{-1}\tan^3\theta+\&c):$$

hence, equating possible and impossible parts of this expression,

$$\cos n\theta = \cos^n\theta \left\{1 - \frac{n(n-1)}{1.2}\tan^2\theta + \frac{n(n-1)(n-2)(-3)}{1.2.3.4}\tan^4\theta - \&c\right\},$$

$$\sin n\theta = \cos^n\theta \left\{n\tan\theta - \frac{n(n-1)(n-2)}{1.2.3}\tan^3\theta + \&c\right\}.$$

Hence also $\tan n\theta = \dfrac{\sin n\theta}{\cos n\theta}$ may be found in terms of tan.

6. EULER'S SERIES. *To shew that*

$$\cos\alpha = 1 - \frac{\alpha^2}{1.2} + \frac{\alpha^4}{1.2.3.4} - \&c, \quad \sin\alpha = \alpha - \frac{\alpha^3}{1.2.3} + \&c.$$

In the results of [5], put $n\theta = \alpha$, or $n = \dfrac{\alpha}{\theta}$;

$$\therefore \cos\alpha = \cos^n\theta \left\{1 - \frac{\alpha(\alpha-\theta)}{1.2}\left(\frac{\tan\theta}{\theta}\right)^2 + \frac{\alpha(\alpha-\theta)(\alpha-2\theta)(\alpha-3\theta)}{1.2.3.4}\left(\frac{\tan\theta}{\theta}\right)^4 - \&c\right\},$$

$$\sin\alpha = \cos^n\theta \left\{\alpha\left(\frac{\tan\theta}{\theta}\right) - \frac{\alpha(\alpha-\theta)(\alpha-2\theta)}{1.2.3}\left(\frac{\tan\theta}{\theta}\right)^3 + \&c\right\}.$$

Now, while α remains unaltered, suppose n to be increased, and θ to be diminished, indefinitely; then $\left(\dfrac{\tan\theta}{\theta}\right)$ tends to 1 as its limiting value, and

$$\cos^n\theta = (1-\sin^2\theta)^{\frac{n}{2}} = 1 - \frac{n}{2}\sin^2\theta + \&c = 1 - \frac{\alpha\theta}{2}\left(\frac{\sin\theta}{\theta}\right)^2 + \&c.$$

and, therefore, tends also to 1 as its Limit; hence ultimately, when n is made infinite and θ vanishes, we have

$$\cos\alpha = 1 - \frac{\alpha^2}{1.2} + \frac{\alpha^4}{1.2.3.4} - \&c, \quad \sin\alpha = \alpha - \frac{\alpha^3}{1.2.3} + \frac{\alpha^5}{1.2.3.4.5} - \&c.$$

From these results we may obtain, by common division,

$$\tan\alpha = \alpha + \frac{\alpha^3}{3} + \frac{2\alpha^5}{15} + \&c,$$

where, however, the Law of the series is not evident.

7. *To obtain exponential expressions for* $\sin\alpha$, $\cos\alpha$, *and* $\tan\alpha$.

Since (146) $e^x = 1 + x + \dfrac{x^2}{1.2} + \dfrac{x^3}{1.2.3} + \dfrac{x^4}{1.2.3.4} + \&c,$

$$\therefore e^{\alpha\sqrt{-1}} = 1 + \alpha\sqrt{-1} - \frac{\alpha^2}{1.2} - \frac{\alpha^3\sqrt{-1}}{1.2.3} + \frac{\alpha^4}{1.2.3.4} + \&c,$$

$$e^{-\alpha\sqrt{-1}} = 1 - \alpha\sqrt{-1} - \frac{\alpha^2}{1.2} + \frac{\alpha^3\sqrt{-1}}{1.2.3} + \frac{\alpha^4}{1.2.3.4} - \&c:$$

hence $e^{a\sqrt{-1}}+e^{-a\sqrt{-1}}=2\left(1-\frac{a^2}{1.2}+\frac{a^4}{1.2.3.4}-\&c\right)=2\cos a$,

$$e^{a\sqrt{-1}}-e^{-a\sqrt{-1}}=2\sqrt{-1}\left(a-\frac{a^3}{1.2.3}+\&c\right)=2\sqrt{-1}\sin a;$$

and, therefore, $\cos a=\frac{1}{2}(e^{a\sqrt{-1}}+e^{-a\sqrt{-1}})$, $\sqrt{-1}\sin a=\frac{1}{2}(e^{a\sqrt{-1}}-e^{-a\sqrt{-1}})$,

and $\therefore$ also $\sqrt{-1}\tan a=\frac{e^{a\sqrt{-1}}-e^{-a\sqrt{-1}}}{e^{a\sqrt{-1}}+e^{-a\sqrt{-1}}}=\frac{e^{2a\sqrt{-1}}-1}{e^{2a\sqrt{-1}}+1}$.

COR. Observe that

$$\cos a+\sqrt{-1}\sin a=e^{a\sqrt{-1}},\ \cos a-\sqrt{-1}\sin a=e^{-a\sqrt{-1}}.$$

8. GREGORY'S SERIES. *To shew that*

$$a=\tan a-\tfrac{1}{3}\tan^3 a+\tfrac{1}{5}\tan^5 a-\&c.$$

From [7] we get

$$e^{2a\sqrt{-1}}=\frac{1+\sqrt{-1}\tan a}{1-\sqrt{-1}\tan a},\text{ or } 2a\sqrt{-1}=\log\frac{1+\sqrt{-1}\tan a}{1-\sqrt{-1}\tan a}:$$

$=2\{\sqrt{-1}\tan a+\frac{1}{3}(\sqrt{-1}\tan a)^3+\frac{1}{5}(\sqrt{-1}\tan a)^5+\&c\}$ by (150)

$=2\sqrt{-1}\{\tan a-\frac{1}{3}\tan^3 a+\frac{1}{5}\tan^5 a-\&c.$

$\therefore a=\tan a-\frac{1}{3}\tan^3 a+\frac{1}{5}\tan^5 a-\&c.$

N.B. Observe that a is here the circular measure of the *least* angle which has the given tangent: for $\frac{a}{\tan a}=1-\frac{1}{3}\tan^2 a+\&c$, where the value of the right side tends to 1 as $\tan a$ tends to *zero;* and this is true also of the left side, if a tends to zero, that is, if a (whether positive or negative) be less than 90°, and so be reduced also to zero, when $\tan a=0$. The same remark applies to the series for the sine, cosine, and tangent in [6].

COR. If $\tan a=x$, and, therefore, $a=\tan^{-1}x$, the preceding result may be written

$$\tan^{-1}x=x-\tfrac{1}{3}x^3+\tfrac{1}{5}x^5-\&c.$$

9. Hence we may arrive at the value of π: for $\tan\frac{1}{4}\pi=1$, and

$$\therefore\frac{1}{4}\pi=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\&c=\frac{2}{1.3}+\frac{2}{5.7}+\frac{2}{9.11}+\&c.$$

Again, if $\tan a=\frac{1}{2}$ and $\tan\beta=\frac{1}{3}$, then (86 Ex. 2) $\tan(a+\beta)=1$;

$\therefore\frac{1}{4}\pi=a+\beta=\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}$

$$=\left\{\frac{1}{2}-\frac{1}{3}\cdot\frac{1}{2^3}+\frac{1}{5}\cdot\frac{1}{2^5}-\&c\right\}+\left\{\frac{1}{3}-\frac{1}{3}\cdot\frac{1}{3^3}+\frac{1}{5}\cdot\frac{1}{3^5}-\&c\right\}$$

$$=\left(\frac{1}{2}+\frac{1}{3}\right)-\frac{1}{3}\left(\frac{1}{2^3}+\frac{1}{3^3}\right)+\frac{1}{5}\left(\frac{1}{2^5}+\frac{1}{3^5}\right)-\&c.$$

But the above series do not converge so rapidly as the following, which is very convenient for the calculation of π.

10. MACHIN'S SERIES. *To shew that*

$$\tfrac{1}{4}\pi = 4\tan^{-1}\tfrac{1}{5} - \tan^{-1}\tfrac{1}{239}.$$

$$\tan^{-1}1 - \tan^{-1}\frac{1}{5} = \tan^{-1}\frac{1-\frac{1}{5}}{1+\frac{1}{5}} = \tan^{-1}\frac{2}{3}$$

$$\tan^{-1}\frac{2}{3} - \tan^{-1}\frac{1}{5} = \tan^{-1}\frac{\frac{2}{3}-\frac{1}{5}}{1+\frac{2}{15}} = \tan^{-1}\frac{7}{17}$$

$$\tan^{-1}\frac{7}{17} - \tan^{-1}\frac{1}{5} = \tan^{-1}\frac{\frac{7}{17}-\frac{1}{5}}{1+\frac{7}{85}} = \tan^{-1}\frac{9}{46}$$

$$\tan^{-1}\frac{9}{46} - \tan^{-1}\frac{1}{5} = \tan^{-1}\frac{\frac{9}{46}-\frac{1}{5}}{1+\frac{9}{230}} = \tan^{-1}\frac{-1}{239};$$

therefore, adding these, we get

$$\tan^{-1}1 - 4\tan^{-1}\tfrac{1}{5} = \tan^{-1}(-\tfrac{1}{239}) = -\tan^{-1}\tfrac{1}{239}, \text{ or } \tfrac{1}{4}\pi = 4\tan^{-1}\tfrac{1}{5} - \tan^{-1}\tfrac{1}{239}$$

11. Hence $\frac{\pi}{4} = 4\{\frac{1}{5} - \frac{1}{3}\times\frac{1}{5^3} + \frac{1}{5}\times\frac{1}{5^5} - \frac{1}{7}\times\frac{1}{5^7} + \&c\} - \{\frac{1}{239} - \&c\}$,

from which the value of π may be calculated as follows:

$\frac{1}{5} = .2$

$\frac{1}{5}\times\frac{1}{5^5} = \frac{1}{5^6} = \frac{2^6}{10^6} = \frac{64}{10^6} = .000064$

$.200064$

subtract $.0026684$

$.1973956$

4

$.7895824$

subtract $\frac{1}{3}\frac{1}{9} = .0041841$

$.78539\,83$

4

$3.14159|32 = \pi$, accurately to five places.

$\frac{1}{3}\times\frac{1}{5^3} = \frac{2^3}{3\times10^3} = \frac{.008}{3} = .0026666$

$\frac{1}{7}\times\frac{1}{5^7} = \frac{2^7}{7\times10^7} = \frac{.0000128}{7} = .0000018$

$.0026684$

CHAPTER II.

ON CERTAIN OTHER TRIGONOMETRICAL SERIES.

12. *To investigate Euler's series for* $\sin\theta$ *and* $\cos\theta$ *in terms of* θ, *without the aid of Demoivre's Theorem.*

Since $\sin(-\theta) = -\sin\theta$, whereas $\cos(-\theta) = \cos\theta$, we see at once that the series for $\sin\theta$ in terms of θ can only contain *odd* powers of θ, and the series for $\cos\theta$ only *even* powers of θ; since the former will be changed in sign but not in value, and the latter will not be changed at all, by writing in it $-\theta$ for θ.

Also when $\theta = 0$, $\sin\theta = 0$, and $\cos\theta = 1$: hence every term in the series for $\sin\theta$ must contain θ, but in that for $\cos\theta$ there must be a term 1, not containing θ.

Assume then

$$\sin\theta = a_1\theta + a_3\theta^3 + a_5\theta^5 + \&c, \quad \cos\theta = 1 + a_2\theta^2 + a_4\theta^4 + \&c.$$

Hence $\frac{\sin\theta}{\theta} = a_1 + a_3\theta^2 + a_5\theta^4 + \&c$, or, putting $\theta = 0$, $\frac{\sin\theta}{\theta} = 1 = a_1$;

putting this value for a_1, and adding the two series, we have

$$1 + \theta + a_2\theta^2 + a_3\theta^3 + a_4\theta^4 + a_5\theta^5 + \&c = \sin\theta + \cos\theta.$$

In this result write $\theta + \phi$ for θ; then

$$1 + (\theta+\phi) + a_2(\theta+\phi)^2 + a_3(\theta+\phi)^3 + a_4(\theta+\phi)^4 + \&c.$$
$$= \sin(\theta+\phi) + \cos(\theta+\phi) = \cos\phi(\cos\theta + \sin\theta) + \sin\phi(\cos\theta - \sin\theta)$$
$$= (1 + a_2\phi^2 + a_4\phi^4 + \&c)\ (1 + \theta + a_2\theta^2 + a_3\theta^3 + a_4\theta^4 + \&c)$$
$$+ (\phi + a_3\phi^3 + a_5\phi^5 + \&c)\ (1 - \theta + a_2\theta^2 - a_3\theta^3 + a_4\theta^4 - \&c):$$

whence, equating coefficients of $\phi 0$, $\phi\theta^2$, $\phi\theta^3$, &c, we get

$$2a_2 = -1,\ 3a_3 = a_2,\ 4a_4 = -a_3,\ \&c,$$

$$\text{or } a_2 = -\frac{1}{1.2},\ a_3 = -\frac{1}{1.2.3},\ a_4 = \frac{1}{1.2.3.4},\ \&c.$$

and, therefore,

$$\sin\theta = \theta - \frac{\theta^3}{1.2.3} + \frac{\theta^5}{1.2.3.4.5} - \&c,\quad \cos\theta = 1 - \frac{\theta^2}{1.2} + \frac{\theta^4}{1.2.3.4} - \&c.$$

13. *If* $2\cos\alpha = x + \frac{1}{x}$, *then* $2\sqrt{-1}\sin\alpha = x - \frac{1}{x}$,

and also $2\cos m\alpha = x^m + \frac{1}{x^m}$, $2\sqrt{-1}\sin m\alpha = x^m - \frac{1}{x^m}$,

For, if $2\cos\alpha = x + \frac{1}{x}$, then $4\cos^2\alpha = x^2 + 2 + \frac{1}{x^2} = 4 - 4\sin^2\alpha$;

$$\therefore -4\sin^2\alpha = x^2 - 2 + \frac{1}{x^2}, \text{ and } 2\sqrt{-1}\sin\alpha = x - \frac{1}{x};$$

hence also $\cos\alpha + \sqrt{-1}\sin\alpha = x$, $\cos\alpha - \sqrt{-1}\sin\alpha = \frac{1}{x}$,

and [3] $\cos m\alpha + \sqrt{-1}\sin m\alpha = x^m$, $\cos m\alpha - \sqrt{-1}\sin m\alpha = \frac{1}{x^m}$;

$$\text{therefore, } 2\cos m\alpha = x^m + \frac{1}{x^m},\ 2\sqrt{-1}\sin m\alpha = x^m - \frac{1}{x^m}.$$

It is obvious from [7 Cor.] that the quantity x here introduced is no other than $e^{\alpha\sqrt{-1}}$, but it is often more convenient to use x than $e^{\alpha\sqrt{-1}}$.

Ex. If $2\cos\alpha = x + \frac{1}{x} = x + x^{-1}$, we have

$$2\cos\tfrac{1}{2}\alpha = x^{\frac{1}{2}} + x^{-\frac{1}{2}},\ 2\sqrt{-1}\sin 3\alpha = x^3 - x^{-3}, \text{ &c.}$$

14. *If* $2\cos\alpha = x + x^{-1}$, *and* $2\cos\beta = y + y^{-1}$, *then* $2\cos(m\alpha + n\beta) = x^m y^n + x^{-m}y^{-n}$.

For $2\cos(m\alpha + n\beta)$

$= 2(\cos m\alpha\cos n\beta - \sin m\alpha\sin n\beta)$

$= \frac{1}{2}(2\cos m\alpha . 2\cos n\beta + 2\sqrt{-1}\sin m\alpha . 2\sqrt{-1}\sin n\beta)$

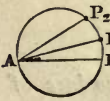
$= \frac{1}{2}\{(x^m + x^{-m})(y^n + y^{-n}) + (x^m - x^{-m})(y^n - y^{-n})\}$ by [13]

$= x^m y^n + x^{-m}y^{-n}$.

This includes the proof that $2\cos(m\alpha - n\beta) = x^m y^{-n} + x^{-m}y^n$, and also, by [13], that $2\sqrt{-1}\sin(m\alpha + n\beta) = x^m y^n - x^{-m}y^{-n}$, &c: and the same reasoning may be extended to any number of angles.

Ex. If $2\cos\gamma = z + z^{-1}$, then $2\cos(\alpha - 2\beta + \frac{1}{3}\gamma) = xy^{-2}z^{\frac{1}{3}} + x^{-1}y^2 z^{-\frac{1}{3}}$.

15. VIETA'S *Property of Chords.*



Let AB be the diameter of a circle, and BP_1, P_1P_2, P_2P_3, &c. circular arcs, each equal to θ; and join AP_1, AP_2, &c: then, if (to radius *unity*) the first of these lines AP_1 be taken $= x + \frac{1}{x}$, the n^{th} line AP_n will $= x^n + \frac{1}{x^n}$.

For $AP_1 = \text{chd}(\pi - \theta) = 2\sin\frac{1}{2}(\pi - \theta) = 2\cos\frac{1}{2}\theta = x + \frac{1}{x}$, by hypothesis;

and $AP_n = \text{chd}(\pi - n\vartheta) = 2\sin\frac{1}{2}(\pi - n\vartheta) = 2\cos\frac{1}{2}n\vartheta = x^n + \frac{1}{x^n}$.

16. *To express* $\cos^n\theta$ *in terms of the cosines of multiples of* θ, *when* n *is a positive integer.*

Put $2\cos\theta = x + x^{-1}$: then we have $(2\cos\theta)^n = (x+x^{-1})^n$

$$= x^n + nx^{n-2} + \frac{n(n-1)}{1.2}x^{n-4} + \&c + \frac{n(n-1)}{1.2}x^{-(n-4)} + nx^{-(n-2)} + x^{-n} \ldots (\alpha)$$

$$= (x^n + x^{-n}) + n(x^{n-2} + x^{-(n-2)}) + \frac{n(n-1)}{1.2}(x^{n-4} + x^{-(n-4)}) + \&c.$$

$$= 2\cos n\theta + n\{2\cos(n-2)\theta\} + \frac{n(n-1)}{1.2}\{2\cos(n-4)\theta\} + \&c:$$

whence, dividing each side by 2, we get

$$2^{n-1}\cos^n\theta = \cos n\theta + n\cos(n-2)\theta + \frac{n(n-1)}{1.2}\cos(n-4)\theta + \&c. \ldots (\beta)$$

If n be even, there will be an odd middle term in (α), namely, the $(\frac{1}{2}n+1)^{\text{th}}$, whose value will be

$$\frac{n(n-1)\ldots(n-\frac{1}{2}n+1)}{1.2\ldots\frac{1}{2}n}x^{\frac{n}{2}}x^{-\frac{n}{2}} = \frac{n(n-1)\ldots(\frac{1}{2}n+1)}{1.2\ldots\frac{1}{2}n};$$

and so the last term of (β) will be $\frac{1}{2}\cdot\frac{n(n-1)\ldots(\frac{1}{2}n+1)}{1.2\ldots\frac{1}{2}n}$

If n be odd, there will be two middle terms in (α), namely, the $\{\frac{1}{2}(n-1)+1\}^{\text{th}}$ and $\{\frac{1}{2}(n+1)+1\}^{\text{th}}$, whose coefficients, however, will be equal, one of these containing the factor $x^{\frac{1}{2}(n+1)}x^{-\frac{1}{2}(n-1)}$ or x, and the other, the factor $x^{\frac{1}{2}(n-1)}x^{-\frac{1}{2}(n+1)}$ or x^{-1}: hence their sum will be

$$\frac{n(n-1)\ldots\{n-\frac{1}{2}(n-1)+1\}}{1.2\ldots\frac{1}{2}(n-1)}(x+x^{-1}) = \frac{n(n-1)\ldots\{\frac{1}{2}(n+3)\}}{1.2\ldots\frac{1}{2}(n-1)}(2\cos\theta);$$

and the last term of (β) will now be $\frac{n(n-1)\ldots\{\frac{1}{2}(n+3)\}}{1.2\ldots\frac{1}{2}(n-1)}\cos\theta.$

Ex. 1. $2^3\cos^4\theta = \cos 4\theta + 4\cos 2\theta + 3.$

Ex. 2. $2^4\cos^5\theta = \cos 5\theta + 5\cos 3\theta + 10\cos\theta.$

17. *To express* $\sin^n\theta$ *in terms of sines or cosines of multiples of* θ, *when* n *is a positive integer.*

Put $2\cos\theta = x + x^{-1}$;

then $2\sqrt{-1}\sin\theta = x - x^{-1}$, and $(-1)^{\frac{n}{2}}(2\sin\theta)^n = (x-x^{-1})^n$

$$= x^n - nx^{n-2} + \frac{n(n-1)}{1.2}x^{n-4} - \&c. + \frac{n(n-1)}{1.2}(-x)^{-(n-4)} + n(-x)^{-(n-2)} + (-x)^{-n}.$$

(i) If n be even, then the above will become

$$(-1)^{\frac{1}{2}n}(2\sin\theta)^n=(x^n+x^{-n})-n(x^{n-2}+x^{-(n-2)})+\&c.$$
$$=2\cos n\theta-n\{2\cos(n-2)\theta\}+\&c;$$

$$\therefore(-1)^{\frac{1}{2}n}.2^{n-1}\sin^n\theta=\cos n\theta-n\cos(n-2)\theta+\frac{n(n-1)}{1.2}\cos(n-4)\theta-\&c,$$

where the last term, as in [16], will be

$$\frac{1}{2}.\frac{n(n-1)\ldots(\frac{1}{2}n+1)}{1.2\ldots\frac{1}{2}n}x^{\frac{n}{2}}(-x)^{-\frac{n}{2}}=(-1)^{\frac{n}{2}}\frac{1}{2}.\frac{n(n-1)\ldots(\frac{1}{2}n+1)}{1.2\ldots\frac{1}{2}n}.$$

(ii) If n be odd, then the above will become

$$(-1)^{\frac{1}{2}n}(2\sin\theta)^n=(x^n-x^{-n})-n(x^{n-2}-x^{-(n-2)})+\&c.$$
$$=2\sqrt{-1}\sin n\theta-n\{2\sqrt{-1}\sin(n-2)\theta\}+\&c.$$

$$\therefore(-1)^{\frac{1}{2}(n-1)}2^{n-1}\sin^n\theta=\sin n\theta-n\sin(n-2)\theta+\frac{n(n-1)}{1.2}\sin(n-4)\theta-\&c,$$

where the last term will be

$$\frac{1}{2}.\frac{n(n-1)\ldots\frac{1}{2}(n+3)}{1.2\ldots\frac{1}{2}(n-1)}\{x^{\frac{n+1}{2}}(-x)^{-\frac{n-1}{2}}+x^{\frac{n-1}{2}}(-x)^{-\frac{n+1}{2}}\}$$
$$=(-1)^{\frac{n-1}{2}}.\frac{1}{2}.\frac{n(n-1)\ldots\frac{1}{2}(n+3)}{1.2\ldots\frac{1}{2}(n-1)}(x-x^{-1})$$
$$=(-1)^{\frac{n-1}{2}}.\frac{n(n-1)\ldots\frac{1}{2}(n+3)}{1.2\ldots\frac{1}{2}(n-1)}\sin\theta.$$

Ex. 1. $-2^5\sin^6\theta=\sin6\theta-6\sin4\theta+15\sin2\theta-10.$

Ex. 2. $+2^4\sin^5\theta=\sin5\theta-5\sin3\theta+10\sin\theta.$

N.B. For the complete forms of the above series for $\cos^n\theta$ and $\sin^n\theta$, for *all* values of n, reference may be made to Peacock's *Algebra*, Vol. II, Artt. 1055—1060.

18. *To express* $\cos n\theta$ *in descending powers of* $\cos\theta$, *when* n *is a positive integer.*

Put $2\cos\theta=x+\frac{1}{x}=y$, suppose; $\therefore\ 2\cos n\theta=x^n+\frac{1}{x^n}$:

now $(1-xz)(1-\frac{z}{x})=1-(x+\frac{1}{x})z+z^2=1-yz+z^2=1-z(y-z)$;

whence, taking logarithms on both sides, we get

$$\log(1-xz)+\log(1-\frac{z}{x})=\log\{1-z(y-z)\},\text{ or}$$

$$xz+\frac{x^2z^2}{2}+\&c+(\frac{z}{x}+\frac{z^2}{2x^2}+\&c)=z(y-z)+\frac{z^2(y-z)^2}{2}+\&c+\frac{z^n(y-z)^n}{n}+\&c.$$

Now the coefficient of z^n in $\frac{z^{n-r}(y-z)^{n-r}}{n-r}$ is

$$\frac{(-1)^r}{n-r} \cdot \frac{(n-r)\ (n-r-1) \dots (n-2r+1)}{1.2 \dots r} y^{n-2r};$$

$\therefore$ equating coefficients of z^n on both sides of the above equation,

$$\frac{1}{n}\left(x^n+\frac{1}{x^n}\right)=\Sigma\left\{\frac{(-1)^r}{n-r} \cdot \frac{(n-r)\ (n-r-1) \dots (n-2r+1)}{1.2 \dots r} y^{n-2r}\right\}, \text{ or}$$

$$2\cos n\theta=n\,\Sigma\left\{\frac{(-1)^r}{n-r} \cdot \frac{(n-r)\ (n-r-1) \dots (n-2r+1)}{1.2 \dots r} (2\cos\theta)^{n-2r}\right\},$$

where r is to be taken from 0 to $\frac{1}{2}n$ if n is even, or to $\frac{1}{2}(n-1)$ if n is odd.

Hence $2\cos n\theta$

$$=(2\cos\theta)^n-n\,(2\cos\theta)^{n-2}+\frac{n\,(n-3)}{1.2}(2\cos\theta)^{n-4}-\&c.$$

19. *To express* $\sin n\theta$ *and* $\cos n\theta$ *in ascending powers of* $\sin\theta$ *and* $\cos\theta$, *of all values of* n.

Assume $\sin n\vartheta=A_0+A_1\sin\theta+A_2\sin^2\theta+\&c+A_p\sin^p\theta+\&c\dots$; (α) then, writing $\theta+h$ for θ, we have

$$\sin n(\theta+h)=A_0+A_1\sin(\theta+h)+A_2\sin^2(\theta+h)+\&c.$$
$$=A_0+A_1\{\sin\theta\,(1-\tfrac{1}{2}h^2)+h\cos\theta\}+\&c,$$

(expanding $\cos h$ and $\sin h$ by [6], and retaining only throughout the investigation the terms involving h and h^2,)

$$=A_0+A_1(\sin\theta+h\cos\vartheta-\tfrac{1}{2}h^2\sin\theta)+A^2(\sin\theta+h\cos\theta-\tfrac{1}{2}h^2\sin\theta)^2+\&c,$$
$$+A_p(\sin\theta+h\cos\theta-\tfrac{1}{2}h^2\sin\theta)^p+\&c.$$
$$=A_0+A_1(\sin\theta+h\cos\theta-\tfrac{1}{2}h^2\sin\theta)$$
$$+A_2(\sin^2\theta+2h\sin\theta\cos\theta-h^2\sin^2\theta+h^2\cos^2\theta)+\&c$$
$$+A_p\{\sin^p\theta+ph\sin^{p-1}\theta\cos\vartheta-\tfrac{1}{2}ph^2\sin^p\theta+\tfrac{1}{2}p(p-1)h^2\sin^{p-2}\theta\cos^2\theta\}+\&c\cdot$$

but $\sin n(\theta+h)=\sin(n\vartheta+nh)=\sin n\vartheta\cos nh+\cos n\vartheta\sin nh$
$$=\sin n\theta\,(1-\tfrac{1}{2}n^2h^2)+nh\cos n\theta+\&c:$$

therefore, equating powers of h, and $\frac{1}{2}h^2$, we get

$$n\cos n\theta=(A_1+2A_2\sin\theta+\&c+pA_p\sin^{p-1}\theta+\&c)\cos\theta\dots(\beta)$$
$$n^2\sin n\vartheta=A_1\sin\theta+2A_2\sin^2\theta+\&c+pA_p\sin^p\theta+\&c$$
$$-\cos^2\theta\{2.1A_2+\&c+p\,(p-1)\,A_p\sin^{p-2}\theta+(p+1)\,pA_{p+1}\sin^{p-1}\theta$$
$$+(p+2)\,(p+1)\,A_{p+2}\sin^p\theta+\&c\}\dots\dots(\gamma).$$

In the last result, write $1-\sin^2\theta$ for $\cos^2\theta$, and compare the coefficient of $\sin^p\theta$, with its value in $n^2\times(a)$: thus we get

$$pA_p+p(p-1)A_p-(p+2)(p+1)A_{p+2}=n^2A_p,$$

$$\text{or } A_{p+2}=-\frac{n^2-p^2}{(p+1)(p+2)}A_p;$$

from which, giving p the successive values 0, 2, 4, &c, we get

$$A_2=-\frac{n^2}{1.2}A_0, \qquad A_4=-\frac{(n^2-2^2)}{3.4}A_2=+\frac{n^2(n^2-2^2)}{1.2.3.4}A_0, \text{ &c};$$

and so, giving p the successive values 1, 3, 5, &c, we get

$$A_3=-\frac{n^2-1^2}{2.3}A_1, A_5=-\frac{n^2-3^2}{4.5}A_3=+\frac{(n^2-1^2)(n^2-3^2)}{2.3.4.5}A_1, \text{ &c.}$$

To find A_0 and A_1, put $\theta=r\pi$ in (a) and (β), r being any integer; then

$$A_0=\sin nr\pi,\ A_1=n\frac{\cos nr\pi}{\cos r\pi}=n\frac{\cos\{(n-1)r\pi+r\pi\}}{\cos r\pi}=n\cos(n-1)r\pi:$$

hence (a) $\sin n\theta=(A_0+A_2\sin^2\theta+\&c)+(A_1\sin\theta+A_3\sin^3\theta+\&c)$

$$=\sin nr\pi\left(1-\frac{n^2}{1.2}\sin^2\theta+\frac{n^2(n^2-2^2)}{1.2.3.4}\sin^4\theta-\&c\right)$$

$$+\cos(n-1)r\pi\left(\frac{n}{1}\sin\theta-\frac{n(n^2-1^2)}{1.2.3}\sin^3\theta+\frac{n(n^2-1^2)(n^2-3^2)}{1.2.3.4.5}\sin^5\theta-\&c\right):(\delta)$$

and (β) $n\cos n\theta=(A_1+3A_3\sin^2\theta+\&c)\cos\theta$

$$+(2A_2\sin\theta+4A_4\sin^3\theta+\&c)\cos\theta,$$

$$\text{or } \cos n\theta=\cos(n-1)r\pi\left(1-\frac{n^2-1^2}{1.2}\sin^2\theta+\frac{(n^2-1^2)(n^2-3^2)}{1.2.3.4}\sin^4\theta-\&c\right)\cos\theta$$

$$-\sin nr\pi\left(n\sin\theta-\frac{n(n^2-2^2)}{1.2}\sin^3\theta+\frac{n(n^2-2^2)(n^2-4^2)}{1.2.3.4}\sin^5\theta-\&c\right)\cos\theta.(\varepsilon)$$

The above results are true for all values of n: but if n be an *odd integer*, then $\sin nr\pi=0$, and $\cos(n-1)\,r\pi=1$;

$$\therefore \sin n\theta=n\sin\theta-\frac{n(n^2-1^2)}{1.2.3}\sin^3\theta+\frac{n(n^2-1^2)(n^2-3^2)}{1.2.3.4.5}\sin^5\theta-\&c, n\,odd\,(1)$$

$$\text{and } \cos n\theta=\left(1-\frac{n^2-1^2}{1.2}\sin^2\theta+\frac{(n^2-1^2)(n^2-3^2)}{1.2.3.4}\sin^4\theta-\&c\right)\cos\theta,\ldots\ldots\ldots(2)$$

where the series in each case will terminate, as soon as the factor n^2-n^2 enters the numerator.

If, however, n be an *even integer*, then, as before, one line in the values of $\sin n\theta$ and $\cos n\theta$ will disappear, but the series in the other line will never terminate, the quantities 1^2, 3^2, &c. in the numerator being all *odd*, and n^2 even.

Exactly in the same manner, by assuming

$$\cos n\theta = A_0 + A_1 \sin\theta + A_2 \sin^2\theta + \&c,$$

we may shew that

$$\cos n\theta = \cos nr\pi \left(1 - \frac{n^2}{1.2}\sin^2\theta + \frac{n^2(n^2-2^2)}{1.2.3.4}\sin^4\theta - \&c\right)$$

$$-\sin(n-1)r\pi \left(n\sin\theta - \frac{n(n^2-1^2)}{1.2.3}\sin^3\theta + \frac{n(n^2-1^2)(n^2-3^2)}{1.2.3.4.5}\sin^5\theta - \&c\right);$$

$$\text{and } \sin n\theta = \cos nr\pi \left(n\sin\theta - \frac{n(n^2-2^2)}{1.2.3.4}\sin^3\theta + \&c\right)\cos\theta$$

$$+\sin(n-1)r\pi \left(1 - \frac{n^2-1^2}{1.2}\sin^2\theta + \frac{(n^2-1^2)(n^2-3^2)}{1.2.3.4}\sin^4\theta - \&c\right)\cos\theta.$$

If n be an integer the second line in each case disappears; and, if n be also even, the first line terminates, and we have

$$\cos n\theta = 1 - \frac{n^2}{1.2}\sin^2\theta + \frac{n^2(n^2-2^2)}{1.2.3.4}\sin^4\theta - \&c, \; n \textit{ even}, \quad \ldots\ldots (3)$$

$$\sin n\theta = \left(n\sin\theta - \frac{n(n^2-2^2)}{1.2.3.4}\sin^3\theta + \&c\right)\cos\theta, \quad \ldots\ldots (4)$$

Lastly, writing $\frac{1}{2}\pi - \theta$ for θ in the four above results, we get

$$(-1)^{\frac{1}{2}(n-1)}\cos n\theta = n\cos\theta - \frac{n(n^2-1^2)}{1.2.3}\cos^3\theta + \&c \; (n \textit{ odd}) \quad \ldots\ldots (5)$$

$$(-1)^{\frac{1}{2}(n-1)}\sin n\theta = \left(1 - \frac{n^2-1^2}{1.2}\cos^2\theta + \frac{(n^2-1^2)(n^2-3^2)}{1.2.3.4}\cos^4\theta - \&c\right)\sin\theta \quad \ldots\ldots (6)$$

$$(-1)^{\frac{1}{2}n}\cos n\theta = 1 - \frac{n^2}{1.2}\cos^2\theta + \frac{n^2(n^2-2^2)}{1.2.3.4}\cos^4\theta - \&c \; (n \textit{ even}) \ldots (7)$$

$$(-1)^{\frac{1}{2}n+1}\sin n\theta = \left(n\cos\theta - \frac{n(n^2-2^2)}{1.2.3.4}\cos^3\theta + \&c\right)\sin\theta \quad \ldots\ldots (8)$$

N.B. The Student, who is acquainted with the Differential Calculus, will see that equations (β) and (γ) may be obtained at once by merely twice differentiating (α) with respect to θ; and, (δ) being thus obtained, (ϵ) also follows from it by differentiation: and, similarly, for the latter set of equations.

CHAPTER III.

ON THE SOLUTION OF EQUATIONS BY TRIGONOMETRY.

20. *To solve a quadratic equation by Trigonometry.*

(i) Let $x^2+px-q=0$; then $x=-\frac{1}{2}p\pm\frac{1}{2}\sqrt{(p^2+4q)}$:

Put $4q=p^2\cot^2\theta$, or $\frac{p}{2\sqrt{q}}=\cot\theta$, which is always possible, since $\cot\theta$ may have any sign or magnitude: then

$x=-\sqrt{q}\,(\cot\theta)\pm\operatorname{cosec}\theta))=-\sqrt{q}\cot\frac{1}{2}\theta$ or $-\sqrt{q}\tan\frac{1}{2}\theta$.

Hence $\mathrm{L}\cot\theta=10+\log p-\log 2-\frac{1}{2}\log q$;

$\log(-x)=\frac{1}{2}\log q+\mathrm{L}\cot\frac{1}{2}\theta-10$, or $=\frac{1}{2}\log q+\mathrm{L}\tan\frac{1}{2}\theta-10$

The same reasoning will serve for the equation $x^2-px-q=0$.

(ii) Let $x^2+px+q=0$; then $x=-\frac{1}{2}p\pm\frac{1}{2}\sqrt{(p^2-4q)}$:

and here, if the roots are possible, or $p^2>4q$, put $4q=p^2\sin^2\theta$, which is always possible, since $p>2\sqrt{q}$, and, therefore, $\sin\theta<1$; then

$x=-\frac{1}{2}p\pm\frac{1}{2}\sqrt{(p^2-p^2\sin^2\theta)}=-\frac{1}{2}p\{1\pm\cos\theta\}=-p\cos^2\frac{1}{2}\theta$, or $p-\sin^2\frac{1}{2}\theta$.

Hence $\mathrm{L}\operatorname{cosec}\theta=10+\log p-\log 2-\frac{1}{2}\log q$;

$\log(-x)=\log p\times 2\,\mathrm{L}\cos\frac{1}{2}\theta-20$, or $=\log p+2\,\mathrm{L}\sin\frac{1}{2}\theta-20$.

But, if $p^2<4q$, put $4q=p^2\sec^2\theta$; then

$x=-\frac{1}{2}p\pm\frac{1}{2}\sqrt{(p^2-p^2\sec^2\theta)}=\frac{1}{2}p\{-1\pm\sqrt{-1}\tan\theta\}$.

The same reasoning will apply to the equation $x^2-px+q=0$.

N.B. The above results are useful, when the quantities p and q happen to be numbers of many figures.

21. *To solve a cubic equation by Trigonometry.*

Let $x^3+ax^2+bx+c=0$ be a cubic equation; by writing $x'-\frac{1}{3}a$ for x, this may be always reduced to the form

$$x'^3-(\tfrac{1}{3}a^2-b)\,x'-(\tfrac{1}{3}ab-\tfrac{2}{27}a^3-c)=0,$$

which wants its second term.

Hence we may assume $x^3-qx-r=0$ as the type of a cubic equation, where q and r may be positive or negative.

For x write $\frac{y}{n}$; then $y^3-qn^2y-rn^3=0$: but we know that

$\cos 3\phi=4\cos^3\phi-3\cos\phi$, or $\cos^3\phi-\frac{3}{4}\cos\phi-\frac{1}{4}\cos 3\phi=0$;

and, if we put $qn^2=\frac{3}{4}$, $rn^3=\frac{1}{4}\cos 3\phi$, these two equations will become identical, that is, we shall have $y=\cos\phi$. Hence we get

$$n^2=\frac{3}{4q},\text{ and }\cos 3\phi=4rn^3=4r\left(\frac{3}{4q}\right)^{\frac{3}{2}}=\frac{r}{2}\sqrt{\frac{27}{q^3}}.$$

Let α be the *least* value of 3ϕ, determined as above: then, since $\cos 3\phi = \cos(2\pi \pm \alpha)$, we get the three values of $\cos\phi$ to be $\cos\frac{1}{3}\alpha$ and $\cos\frac{1}{3}(2\pi \pm \alpha)$; and, therefore, those of $\frac{y}{n}$, or x, will be $2\sqrt{\frac{q}{3}}\cos\frac{\alpha}{3}$, and $2\sqrt{\frac{q}{3}}\cos\frac{2\pi \pm \alpha}{3}$, where $\cos\alpha = \frac{r}{2}\sqrt{\frac{27}{q^3}}$.

Since $\cos\alpha < 1$, we must have $\frac{r}{2}\sqrt{\frac{27}{q^3}} < 1$, or $\frac{r^2}{4} < \frac{q^3}{27}$; that is, the above method can only be applied when *Cardan's* method of solution, which is given below, is found to fail.

22. *To adapt the roots of Cardan's Solution for Logarithmic computation.*

In the equation $x^3 - qx - r = 0$, put $x = y + z$; then we get

$$y^3 + z^3 + (3yz - q)(y + z) - r = 0:$$

assume $3yz - q = 0$, and, therefore, $y^3 + z^3 = r$, where $y^3 z^3 = \frac{1}{27}q^3$: hence y^3 and z^3 are the roots of the quadratic $v^2 - rv + \frac{1}{27}q^3 = 0$, so that

$$y^3 = \frac{r}{2} + \sqrt{(\frac{r^2}{4} - \frac{q^3}{27})},\ z^3 = \frac{r}{2} - \sqrt{(\frac{r^2}{4} - \frac{q^3}{27})},$$

and $$x = y + z = \sqrt[3]{\{\frac{r}{2} + \sqrt{(\frac{r^2}{4} - \frac{q^3}{27})}\}} + \sqrt[3]{\{\frac{r}{2} - \sqrt{(\frac{r^2}{4} - \frac{q^3}{27})}\}}.$$

The above is *Cardan's Solution*, where, however, if $\frac{r^2}{4} < \frac{q^3}{27}$, the roots, though really possible, appear under an impossible form, and the method is said to fail.

(i) To this case the reasoning of the last Article applies; and the roots, there obtained, may be easily derived from the above.

For put $\frac{r^2}{4} = \frac{q^3}{27}\cos^2\alpha$, which is always possible, since $\frac{r^2}{4} < \frac{q^3}{27}$;

then $$x = \sqrt{\frac{q}{3}}\{(\cos\alpha + \sqrt{-1}\sin\alpha)^{\frac{1}{3}} + (\cos\alpha - \sqrt{-1}\sin\alpha)^{\frac{1}{3}}\}$$

$$= 2\sqrt{\frac{q}{3}}\cos\frac{\alpha}{3} \text{ or } 2\sqrt{\frac{q}{3}}\cos\frac{2\pi + \alpha}{3},$$

as appears easily by Demoivre's Theorem.

(ii) If, however, $\frac{r^2}{4} > \frac{q^3}{27}$, then, if the original equation be of the form $x^3 - qx - r = 0$, where the coefficient of x is really negative, put $\frac{r^2}{4} = \frac{q^3}{27}\operatorname{cosec}^2\theta$: then we have

$$x=\sqrt{\frac{q}{3}}\{\sqrt[3]{(\operatorname{cosec}\theta+\cot\theta)}+\sqrt[3]{(\operatorname{cosec}\theta-\cot\theta)}\}$$

$$=\sqrt{\frac{q}{3}}\{\sqrt[3]{\cot\tfrac{1}{2}\theta}+\sqrt[3]{\tan\tfrac{1}{2}\theta}\}=\sqrt{\frac{q}{3}}(\cot\phi+\tan\phi), \text{ if } \tan\phi=\sqrt[3]{\tan\tfrac{1}{2}\theta},$$

$$=\sqrt{\frac{q}{3}}\cdot\frac{1}{\sin\phi\cos\phi}=2\sqrt{\frac{q}{3}}\operatorname{cosec}2\phi.$$

But, if the original equation be of the form $x^3+qx-r=0$, where the coefficient of x is positive, we should get in the same way

$$x=\sqrt[3]{\{\frac{r}{2}+\sqrt{(\frac{r^2}{4}+\frac{q^3}{27})}\}}+\sqrt[3]{\{\frac{r}{2}-\sqrt{(\frac{r^2}{4}+\frac{q^3}{27})}\}};$$

and now put $\frac{r^2}{4}=\frac{q^3}{27}\cot^2\theta$: then, as before, we obtain

$$x=\sqrt{\frac{q}{3}}\{\sqrt[3]{(\cot\theta+\operatorname{cosec}\theta)}+\sqrt[3]{(\cot\theta-\operatorname{cosec}\theta)}\}$$

$$=\sqrt{\frac{q}{3}}(\sqrt[3]{\cot\tfrac{1}{2}\theta}-\sqrt[3]{\tan\tfrac{1}{2}\theta})=\sqrt{\frac{q}{3}}(\cot\phi-\tan\phi)=2\sqrt{\frac{q}{3}}\cot2\phi.$$

23. *To find the roots of* $x^n-1=0$, *and to resolve* x^n-1 *into factors.*

If $x^n-1=0$, then we have $x^n=1=\cos 2r\pi\pm\sqrt{-1}\sin 2r\pi$;

$$\therefore x=\cos\frac{2r\pi}{n}\pm\sqrt{-1}\sin\frac{2r\pi}{n}.$$

From this expression, by giving r all values from 0 to $\frac{1}{2}n$ if n be even, or to $\frac{1}{2}(n-1)$ if n be odd, we shall obtain n values for x, the *first* pair of roots being each $=+1$, in both cases, and the *last* pair being each $=-1$, when n is *even*.

Now, excepting those just mentioned, all these values will be different from one another: for, suppose that

$$\cos\frac{2p\pi}{n}+\sqrt{-1}\sin\frac{2p\pi}{n}=\cos\frac{2q\pi}{n}\pm\sqrt{-1}\sin\frac{2qp}{n}:$$

then we get $\cos\frac{2p\pi}{n}=\cos\frac{2q\pi}{n}$, and $\sin\frac{2p\pi}{n}=\pm\sin\frac{2q\pi}{n}$;

$$\text{hence } \frac{2p\pi}{n}=2m\pi\pm\frac{2q\pi}{n}, \text{ or } \frac{p\mp q}{n}=m, \text{ an integer.}$$

Hence $p\mp q$ is a multiple of n, which is impossible, since p and q are each less than $\frac{1}{2}n$: and, consequently, the n values of x, obtained as above, will be all different from one another.

And, further, we can get no new values for x by taking for r a value r', which is greater than $\frac{1}{2}n$. For any integer r' may be put into the form $mn+r$, where r is not greater than $\frac{1}{2}n$; and

$$\cos\frac{2r'\pi}{n}+\sqrt{-1}\sin\frac{2r'\pi}{n}=\cos(2m\pi\pm\frac{2r\pi}{n})+\sqrt{-1}\sin(2m\pi\pm\frac{2r\pi}{n})$$

$$=\cos\frac{2r\pi}{n}\pm\sqrt{-1}\sin\frac{2r\pi}{n},$$

each of which values is included among those already obtained, since r is not greater than $\frac{1}{2}n$: and similarly for

$$\cos\frac{2r'\pi}{n}-\sqrt{-1}\sin\frac{2r'\pi}{n}.$$

We obtain, therefore, the n roots of the equation $x^n-1=0$ from the above expression, by putting $r=0, 1, 2,$ &c. successively.

Now (i) if n be even, these values of x are

$$+1,\ \cos\frac{2\pi}{n}\pm\sqrt{-1}\sin\frac{2\pi}{n},\ \cos\frac{4\pi}{n}+\sqrt{-1}\sin\frac{4\pi}{n},\ \&c,$$

$$\cos\frac{(n-2)\pi}{n}\pm\sqrt{-1}\sin\frac{(n-2)\pi}{n},\ -1,$$

the first and last values arising when $r=0$ and when $r=\frac{1}{2}n$; hence, by the Theory of Equations, (observing that

$$\{x-(\cos\theta+\sqrt{-1}\sin\theta)\}\ \{x-(\cos\theta-\sqrt{-1}\sin\theta)\}$$
$$=\{(x-\cos\theta)-\sqrt{-1}\sin\theta\}\ \{(x-\cos\theta)+\sqrt{-1}\sin\theta\}$$
$$=(x-\cos\theta)^2+\sin^2\theta=x^2-2x\cos\theta+1,)$$

we have $x^n-1=(x^2-1)\ (x^2-2x\cos\frac{2\pi}{n}+1)\ (x^2-2x\cos\frac{4\pi}{n}+1)\ldots$

So (ii), if n be odd, we shall get

$$(x^n-1)=(x-1)\ (x^2-2x\cos\frac{2\pi}{n}+1)\ (x^2-2x\cos\frac{4\pi}{n}+1)..$$

Cor. Instead of using the double sign, we might solve the above equation as follows:

$x^n=1=\cos 2r\pi+\sqrt{-1}\sin 2r\pi$, and $x=\cos\frac{2r\pi}{n}+\sqrt{-1}\sin\frac{2r\pi}{n}$,

where r may be taken from 1 to n.

These values of x will, of course, be identical with the former: thus, if $r<n$,

$$\cos\frac{2(n-r)\pi}{n}+\sqrt{-1}\sin\frac{2(n-r)\pi}{n}=\cos(2\pi-\frac{r\pi}{n})+\sqrt{-1}\sin(2\pi-\frac{r\pi}{n})$$

$$=\cos\frac{r\pi}{n}-\sqrt{-1}\sin\frac{r\pi}{n},$$

one of the roots which would be found in [23].

Hence the different values of x may be expressed by the series

$$\cos\frac{2\pi}{n}+\sqrt{-1}\sin\frac{2\pi}{n},\ \cos\frac{4\pi}{n}+\sqrt{-1}\sin\frac{4\pi}{n},\ \cos\frac{6\pi}{n}+\sqrt{-1}\sin\frac{6\pi}{n},\ \&c;$$

or, denoting the *first* of them by a, it follows, by Demoivre's Theorem, that they may all be expressed by the series $a,\ a^2,\ a^3 \ldots a^n$ or 1.

Ex. If $x^3-1=0$, the root (a) of x will be

$$\cos\tfrac{2}{3}\pi+\sqrt{-1}\sin\tfrac{2}{3}\pi=-\tfrac{1}{2}+\tfrac{1}{2}\sqrt{3}\sqrt{-1};$$

and the other two roots will be

$$(-\tfrac{1}{2}+\tfrac{1}{2}\sqrt{3}\sqrt{-1})^2=-\tfrac{1}{2}-\tfrac{1}{2}\sqrt{3}\sqrt{-1}, \text{ and } (-\tfrac{1}{2}+\tfrac{1}{2}\sqrt{3}\sqrt{-1})^3=1.$$

24. *To find the roots of* $x^n+1=0$, *and to resolve* x^n+1 *into factors.*

If $x^n+1=0$, then $x^n=-1=\cos(2r+1)\pi\pm\sqrt{-1}\sin(2r+1)\pi$;

and, consequently, $x=\cos\dfrac{(2r+1)\pi}{n}\pm\sqrt{-1}\sin\dfrac{(2r+1)\pi}{n}$,

from which expression we shall get the n different values of x, by putting $r=0,\ 1,\ 2,$ &c. to $\frac{1}{2}n-1$ when n is even, or to $\frac{1}{2}(n-1)$ when n is odd, the last pair in the latter case being each $=-1$.

Hence, (i) if n be even,

$$(x^n+1)=(x^2-2x\cos\frac{\pi}{n}+1)\ (x^2-2x\cos\frac{3\pi}{n}+1)\ldots$$

or, (ii) if n be odd,

$$(x^n+1)=(x+1)\,(x^2-2x\cos\frac{\pi}{n}+1)\ (x^2-2x\cos\frac{3\pi}{n}+1)\ldots$$

where the factor $x+1$ arises from the root -1, obtained by putting $r=\frac{1}{2}(n-1)$.

25. If it be required to resolve x^n-a^n or x^n+a^n into factors, put $x=az$; then

$$x^n-a^n=a^n(z^n-1)=a^n(z^2-1)\,(z^2-2z\cos\frac{2\pi}{n}+1)\ldots \text{if } n \text{ be even}$$

$$=(x^2-a^2)\ (x^2-2ax\cos\frac{2\pi}{n}+a^2)\ldots$$

and so with the other cases.

26. From the preceding results many remarkable factorials may be derived.

Since $x^{2n}-1=(x^2-1)\,(x^2-2x\cos\dfrac{\pi}{n}+1)\ (x^2-2x\cos\dfrac{2\pi}{n}+1\ldots$

the index being even,

$$\therefore x^{2n-2}+x^{2n-4}+\&c+x^2+1=(x^2-2x\cos\frac{\pi}{n}+1)\,(x^2-2x\cos\frac{2\pi}{n}+1)\ldots$$

In this expression put $x=1$; then we get

$$n=(2-2\cos\frac{\pi}{n})(2-2\cos\frac{2\pi}{n})\cdots\cdots(2-2\cos\frac{(n-1)\pi}{n})$$

$$=4\sin^2\frac{\pi}{2n}\cdot 4\sin^2\frac{2\pi}{2n}\cdots\cdots 4\sin^2\frac{(n-1)\pi}{2n},$$

or $$\sqrt{n}=2^{n-1}\cdot\sin\frac{\pi}{2n}\cdot\sin\frac{2\pi}{2n}\cdots\sin\frac{(n-1)\pi}{2n}, \quad \ldots\ldots\ldots\ldots\ldots (1)$$

where we take the positive sign with the root, because each of the above angles being $<\frac{1}{2}\pi$, every sine will be positive.

So also

$$x^{2n+1}-1=(x-1)(x^2-2x\cos\frac{2\pi}{2n+1}+1)(x^2-2x\cos\frac{4\pi}{2n+1}+1)\ldots$$

the index being odd;

$\therefore$ dividing by $x-1$, and putting $x=1$, we get

$$\sqrt{(2n+1)}=2^n\sin\frac{\pi}{2n+1}\cdot\sin\frac{2\pi}{2n+1}\cdots\sin\frac{n\pi}{2n+1}\ldots \quad (2$$

Again, $$x^{2n}+1=(x^2-2x\cos\frac{\pi}{2n}+1)(x^2-2x\cos\frac{3\pi}{2n}+1)\ldots$$

put $x=1$; then $$2=4\sin^2\frac{\pi}{4n}\cdot 4\sin^2\frac{3\pi}{4n}\cdots\cdot 4\sin^2\frac{(2n-1)\pi}{4n},$$

or $$\sqrt{2}=2^n\sin\frac{\pi}{4n}\cdot\sin\frac{3\pi}{4n}\cdots\sin\frac{(2n-1)\pi}{4n}. \quad \ldots\ldots(3)$$

And $$x^{2n+1}+1=(x+1)(x^2-2x\cos\frac{\pi}{2n+1}+1)(x^2-2x\cos\frac{3\pi}{2n+1}+1)\ldots,$$

dividing by $x+1$, and putting $x=1$, we get

$$1=2^n\sin\frac{\pi}{2(2n+1)}\cdot\sin\frac{3\pi}{2(2n+1)}\cdots\sin\frac{(2n-1)\pi}{2(2n+1)}\cdots (4)$$

It is obvious that other results may be obtained in the same way by putting $x=-1$, which would make the factors consist of cosines instead of sines, and so, by division, we might arrive at other similar results, where the factors are tangents.

Thus, in the equation which led to (1), putting $x=-1$, we shall get

$$\sqrt{n}=2^{n-1}\cos\frac{\pi}{2n}\cdot\cos\frac{2\pi}{2n}\cdots\cos\frac{(n-1)\pi}{2n};$$

and therefore also, dividing (1) by this result, we have

$$1=\tan\frac{\pi}{2n}\cdot\tan\frac{2\pi}{2n}\cdots\tan\frac{(n-1)\pi}{2n}\cdots (5)$$

27. *To resolve* $x^{2n}-2x^n\cos\theta+1$ *into quadratic factors.*

Put $x^{2n}-2x^n\cos\theta+1=0$; then $x^{2n}-2x^n\cos\theta+\cos^2\theta=-\sin^2\theta$; hence $x^n=\cos\theta\pm\sqrt{-1}\sin\theta=\cos(2r\pi+\theta)\pm\sqrt{-1}\sin(2r\pi+\theta)$;

$$\therefore x = \cos\frac{2r\pi+\theta}{n} \pm \sqrt{-1}\sin\frac{2r\pi+\theta}{n},$$

where the $2n$ different values of x may be obtained by taking r from 0 to $n-1$.

Hence, as in [23], we get

$$x^{2n} - 2x^n\cos\theta + 1 = (x^2 - 2x\cos\frac{\theta}{n} + 1)\ (x^2 - 2x\cos\frac{2\pi+\theta}{n} + 1)$$
$$\ldots(x^2 - 2x\cos\frac{2(n-1)\pi+\theta}{n} + 1); \ldots\ldots\ldots\ldots (1)$$

whence we get, putting $\frac{x}{a}$ for x, and multiplying each side by a^{2n},

$$x^{2n} - 2a^nx^n\cos\theta + a^{2n} = (x^2 - 2ax\cos\frac{\theta}{n} + a^2)\ (x^2 - 2ax\cos\frac{2\pi+\theta}{n} + a^2)\ldots$$

From the above result, putting $\theta = 0$, we get

$$x^{2n} - 2a^nx^n + a^{2n} = (x^n \sim a^n)^2 = (x^2 - 2ax + a^2)\ (x^2 - 2ax\cos\frac{2\pi}{n} + a^2)\ldots;$$

and, putting $\theta = \pi$,

$$x^{2n} + 2a^nx^n + a^{2n} = (x^n + a^n)^2 = (x^2 - 2ax\cos\frac{\pi}{n} + a^2)\ (x^2 - 2ax\cos\frac{3\pi}{n} + a^2)\ldots$$

28. If we observe that

$$\cos\frac{2(n-1)\pi+\theta}{n} = \cos(2\pi - \frac{2\pi-\theta}{n}) = \cos\frac{2\pi-\theta}{n},\ \&c.$$

the expression (1) in the last article may be put into the form

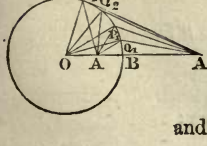
$$x^{2n} - 2x^n\cos\theta + 1$$
$$= (x^2 - 2x\cos\frac{\theta}{n} + 1)\ (x^2 - 2x\cos\frac{2\pi+\theta}{n} + 1)\ (x^2 - 2x\cos\frac{4\pi+\theta}{n} + 1)\ldots,$$

where both signs are to be taken, and there must be altogether n quadratic factors — the last, consequently, being either a *single* or a *double* factor, according as n is even or odd.

Cor. The above expression will be found to include those already found for $x^n \pm 1$; since, by putting $\theta = 0$ and $\theta = \pi$, we may get the resolution into factors of $(x^n \mp 1)^2$.

29. *Cotes's Properties of the Circle.*

Let A be any point in the radius OB, or in the radius OB produced, of a circle whose centre is O; and let the circumference be divided into n equal parts BP_1, P_1P_2, &c, and into $2n$ equal parts BQ_1, Q_1P_1, &c: then



$$AP_1.AP_2.AP_3\ldots\text{to } n \text{ factors} = OA^n \sim OB^n,$$
and $$AQ_1.AQ_2.AQ_3\ldots\text{to } n \text{ factors} = OA^n + OB^n.$$

For let $OA=x$, $OB=a$; then we have

$$AP_1^2=OA^2-2OA.OP_1\cos\frac{2\pi}{n}+OP_1^2=x^2-2ax\cos\frac{2\pi}{n}+a^2, \text{ \&c:}$$

$$\therefore AP_1^2.AP_2^2.AP_3^2\ldots=(x^2-2ax\cos\frac{2\pi}{n}+a^2)\,(x^2-2ax\cos\frac{4\pi}{n}+a^2)\ldots$$

$$=(x^n\sim a^n)^2 \text{ by [27]};$$

or $AP_1.AP_2.AP_3\ldots=OA^n\sim OB^n=OB^n-OA^n$ or OA^n-OB^n, according as A is in OB or OB produced.

Again, $AQ_1.AP_1.AQ_2.AP_2\ldots\ldots=OA^{2n}\sim OB^{2n}$, by the above; and therefore, dividing this by the former result, we get

$$AQ_1.AQ_2.AQ_3\ldots n \text{ factors}=OA^n+OB^n.$$

30. *To resolve $\sin\theta$, $\cos\theta$, $\tan\theta$ into factors of sines, cosines, and tangents.*

In the expression (1) of [27], put $x=1$: then

$$2-2\cos\theta=(2-2\cos\frac{\theta}{n})\,(2-2\cos\frac{2\pi+\theta}{n})\ldots(2-2\cos\frac{2(n-1)\pi+\theta}{n})$$

$$\text{or } 4\sin^2\frac{\theta}{2}=4\sin^2\frac{\theta}{2n}.4\sin^2\frac{2\pi+\theta}{2n}\ldots 4\sin^2\frac{2(n-1)\pi+\theta}{2n};$$

$$\text{whence } \sin\frac{\theta}{2}=2^{n-1}\sin\frac{\theta}{2n}\sin\frac{2\pi+\theta}{2n}\sin\frac{4\pi+\theta}{2n}\ldots\sin\frac{2(n-1)\pi+\theta}{2n},$$

or writing 2θ for θ,

$$\sin\theta=2^{n-1}\sin\frac{\theta}{n}\sin\frac{\pi+\theta}{n}\sin\frac{2\pi+\theta}{n}\ldots\text{to } n \text{ factors}\ldots\ldots(1).$$

N.B. We have taken the sign of the square root (+), which, as will be shewn, is always the case in the above resolution of $\sin\theta$.

Again, in the expression (1) of [27], put $x=-1$;

$$\left.\begin{array}{ll}\text{then} & (n \textit{ even})\ 2-2\cos\\ \text{or} & (n \textit{ odd})\ \ 2+2\cos\theta\end{array}\right\}=(2+2\cos\frac{\theta}{n})(2+2\cos\frac{2\pi+\theta}{n})\ldots;$$

whence, writing 2θ for θ, and proceeding as before, we get

$$\left.\begin{array}{l}(n \textit{ even})\pm\sin\theta\\ \text{or } (n \textit{ odd})\ \pm\cos\theta\end{array}\right\}=2^{n-1}\cos\frac{\theta}{n}\cos\frac{\pi+\theta}{n}\cos\frac{2\pi+\theta}{n}\ldots\text{to } n \text{ factors}\ldots(2).$$

Dividing (1) by (2), we get

$$\left.\begin{array}{l}(n \textit{ even})\pm1\\ (n \textit{ odd})\ \pm\tan\theta\end{array}\right\}=\tan\frac{\theta}{n}\tan\frac{\pi+\theta}{n}\tan\frac{2\pi+\theta}{n}\ldots\text{to } n \text{ factors}\ldots(3).$$

Cor. If in (1) above we write $\frac{1}{2}\pi$ for θ, we get

$$1=2^{n-1}\sin\frac{\pi}{2n}\sin\frac{3\pi}{2n}\sin\frac{5\pi}{2n}\ldots\text{ to } n \text{ factors:}$$

if in (1) we write 0 for θ, we get

$$\sin\theta \div \sin\frac{\theta}{n} = n = 2^{n-1}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\sin\frac{3\pi}{n} \ldots \text{ to } (n-1) \text{ factors :}$$

if in (3) we write $\frac{1}{4}\pi$ for θ, we get, whether n be even or odd,

$$\pm 1 = \tan\frac{\pi}{4n}\tan\frac{5\pi}{4n}\tan\frac{9\pi}{4n} \ldots \text{ to } n \text{ factors.}$$

And many similar formulæ may easily be derived.

31. We shall now shew that in the expression (1) of [30], the (+) sign of the root is always to be taken.

Write $n\theta$ for θ in it: then

$$\sin n\theta = 2^{n-1}\sin\theta\sin(\theta+\frac{\pi}{n})\sin(\theta+\frac{2\pi}{n}) \ldots\ldots \text{to } n \text{ factors}$$

Now (i) let $n\theta = m\pi - \alpha$, where $\alpha < \pi$, and m = zero or any positive integer: (this will include all *positive* values of θ, and all *negative* values between 0 and $-\pi$:) then $\sin n\theta = (-1)^{m-1}\sin\alpha$, and will therefore have the sign of $(-1)^{m-1}$.

The general type of the factors on the right-hand side is

$$\sin(\theta+\frac{r\pi}{n}), \text{ where } r \text{ may range from 0 to } n-1 :$$

now $\sin(\theta+\frac{r\pi}{n}) = \sin\frac{(m+r)\pi-\alpha}{n}$, where, since $\alpha < \pi$, the angle itself is always positive: hence, in the case of m positive, this factor is *positive* or *negative*, alternately, while $\frac{(m+r)\pi-\alpha}{n}$ lies between 0 and π, π and 2π, 2π and 3π, &c, that is, it is *positive*, while $r < (n-m)+\frac{\alpha}{\pi}$, or from $r=0$ to $r=n-m$ inclusive, *negative* while $r > n-m < 2n-m+\frac{\alpha}{\pi}$, or from $r=n-m+1$ to $r=2n-m$, inclusive, &c, and, generally, it has the sign of $(-1)^p$ from $r=pn-m+1$ to $r=(p+1)\,n-m$ inclusive.

Hence (1) if $m<n$, we shall have $n-m+1$ positive factors, (from $r=0$ to $r=n-m$), and (there being n factors altogether) $m-1$ negative; so that the sign of the product will be that of $(-1)^{m-1}$;

(2) if $m=n$, then we shall have only *one* positive factor (when $r=n-m=0$), and $n-1=m-1$ negative factors: so that here too the sign of the product will be that of $(-1)^{m-1}$;

(3) if $m>n$, or if $m=pn+m'$, suppose, (where $m'<n$,) then the lowest value of r ($r=0$) will occur between $pn-m+1$ and $(p+1)n-m$, namely when $r=(pn-m)+m'=0$; hence there will be $\{(p+1)n-m\}-\{(pn-m)+m'\}+1$ or $n-m'+1$ factors with the sign $(-1)^p$, and the remaining $m'-1$ factors with the sign $(-1)^{p+1}$: so that the sign of the product will be that of

$$(-1)^{(n-m'+1)p+(m'-1)(p+1)} \text{ or } (-1)^{np+m'-1}=(-1)^{m-1}, \text{ as before.}$$

Hence, in all cases, when m is *positive*, the sign of the product of the n factors is that of $(-1)^{m-1}$, which is the same as that of $\sin n\theta$.

(ii) Let $n\theta=-m\pi-\alpha$; (this will include all *negative* values of θ, from $-\pi$ downwards:) then $\sin n\theta=(-1)^{m-1}\sin\alpha$, as before; and $\sin(\theta+\frac{r\pi}{n})=\sin\frac{(r-m)\pi-\alpha}{n}$, where, since $\alpha<\pi$, the angle is positive, from $r=m+1$ to $r=n-1$, (its highest value,) and negative for all values of r less than $m+1$. Now, while the angle is positive, its sine is positive while $\frac{(r-m)\pi-\alpha}{n}$ lies between 0 and π, that is, from $r=m+1$ to $r=m+n$, both inclusive; and hence *all* the factors will be positive, for which the angle is positive, namely, from $r=m+1$ to $r=n-1$. But, when the angle is negative, its sine is *negative* or *positive*, alternately, while $\frac{(r-m)\pi-\alpha}{n}$ lies between 0 and $-\pi$, $-\pi$ and -2π, &c, that is, it is *positive* from $r=m$ to $r=m-n+1$, *negative* from $r=m-n$ to $r=m-2n+1$, &c; and, generally, it has the sign of $(-1)^p$ from $r=m-(p-1)n$ to $r=m-pn+1$.

Hence, (1) if $m<n$, we shall have $n-m-1$ factors positive, (from $r=m+1$ to $r=n-1$,) and, consequently, $m+1$ negative factors, so that the sign of the product will be that of $(-1)^{m+1}=(-1)^{m-1}$:

(2) if $m=n$, there will be n negative factors (from $r=m-1$ or $n-1$ (its highest value) to $r=m-n+1=1$,) and *one* positive factor (when $r=m-n=0$); so that here too the sign of the product will be $(-1)^{m-1}$:

(3) if $m>n$, or if $m=pn+m'$, where $m'<n$, then the lowest value of r ($r=0$) will occur between $r=m-pn$, and $r=m-(p+1)n+1$, and will be when $r=m-pn-m'=0$; hence there will be $(m-pn)-(m-pn-m')+1=m'+1$ factors with the sign $(-1)^{p+1}$

and the remaining $n-m'-1$ with the sign $(-1)^p$; so that the sign of the product will be that of

$$(-1)^{(m'+1)(p+1)+(n-m'-1)p} = (-1)^{m'+np+1} = (-1)^{m+1} \text{ or } (-1)^{m-1},$$

as before: that is, in all cases, the product of the n factors will have the same sign as $\sin n\theta$.

Similar reasonings would enable us to determine in what cases the + or − sign is to be used in the resolution of $\cos\theta$ and $\tan\theta$. For example, in [30 Cor.], if $n=1$, then $\tan\frac{1}{4}\pi = +1$, if $n=2$, then $\tan\frac{1}{8}\pi \tan\frac{5}{8}\pi = -1$, if $n=3$, then $\tan\frac{1}{12}\pi \tan\frac{5}{12}\pi \tan\frac{9}{12}\pi = -1$, two tangents in the last case being positive, the other negative.

32. *To resolve $\sin n\theta$ and $\cos n\theta$ into quadratic factors.*

The expression (1) in [30] for $\sin\theta$, (observing that

$$\sin\frac{(n-1)\pi+\theta}{n} = \sin\left(\pi - \frac{\pi-\theta}{n}\right) = \sin\left(\frac{\pi-\theta}{n}\right),$$

may be put into the form

$$\sin\theta = 2^{n-1}\sin\frac{\theta}{n}\sin\frac{\pi\pm\theta}{n}\sin\frac{2\pi\pm\theta}{n}\ldots,$$

where both signs are to be taken, and there are to be n factors altogether, the last being *single* (with the upper sign only) or *double*, according as n is even or odd.

Write $n\vartheta$ for θ, and $2n\alpha$ for π, in the above expression.

(1) If n be odd,

$\sin n\vartheta = 2^{n-1}\sin\vartheta \sin(2\alpha\pm\theta \sin(4\alpha\pm\theta)\ldots\sin\{(n-1)\alpha\pm\theta\}$;

whence, writing $\frac{1}{2}\pi+n\vartheta$ for $n\vartheta$, or $\alpha+\theta$ for θ, we get also

$$\cos n\theta = 2^{n-1}\sin(\alpha+\theta)\sin(\alpha-\theta)\sin(3\alpha+\theta)\sin(3\alpha-\theta)\ldots$$
$$\sin\{(n-2)\alpha-\theta\}\sin(n\alpha+\theta),$$
$$= 2^{n-1}\cos\vartheta \sin(\alpha\pm\theta)\sin(3\alpha\pm\theta)\ldots\sin\{(n-2)\alpha\pm\theta\},$$

since the last factor $\sin(n\alpha+\theta) = \sin(\frac{1}{2}\pi+\theta) = \cos\theta$.

Hence, observing that $\sin(\alpha+\theta)\sin(\alpha-\theta) = \sin^2\alpha - \sin^2\vartheta$, we get

$\sin n\vartheta = 2^{n-1}\sin\vartheta(\sin^2 2\alpha - \sin^2\vartheta)(\sin^2 4\alpha - \sin^2\vartheta)\ldots\{\sin^2(n-1)\alpha - \sin^2\theta\}$.

$\cos n\theta = 2^{n-1}\cos\vartheta(\sin^2\alpha - \sin^2\theta)(\sin^2 3\alpha - \sin^2\vartheta)\ldots\{\sin^2(n-2)\alpha - \sin^2\vartheta\}$

(2) If n be *even*, we have, as before,

$\sin n\vartheta = 2^{n-1}\sin\vartheta\sin(2\alpha\pm\theta)\sin(4\alpha\pm\theta)\ldots\{\sin(n-2)\alpha\pm\theta\}\sin(n\alpha+\theta)$

$= 2^{n-1}\sin\vartheta\cos\vartheta(\sin^2 2\alpha - \sin^2\vartheta)(\sin^2 4\alpha - \sin^2\vartheta)\ldots\{\sin^2(n-2)\alpha - \sin^2\theta\}$;

$$\cos n\theta = 2^{n-1}\sin(\alpha+\theta)\sin(\alpha-\theta)\sin(3\alpha+\theta)\sin(3\alpha-\theta)\ldots$$
$$\sin\{(n-1)\alpha+\theta\}\sin\{(n+1)\alpha+\theta\},$$
$$= 2^{n-1}(\sin^2\alpha - \sin^2\vartheta)(\sin^2 3\alpha - \sin^2\vartheta)\ldots\{\sin^2(n-1)\alpha - \sin^2\theta\},$$

since $\sin\{(n+1)\alpha+\theta\} = \sin(\frac{1}{2}\pi+\alpha+\vartheta) = \sin(\frac{1}{2}\pi-\alpha-\theta) = \sin\{(n-1)\alpha-\theta\}$.

33. We may employ the preceding results to resolve $\sin\theta$ and $\cos\theta$ into quadratic factors in terms of θ.

Thus in the formulæ for $\sin n\theta$, $\cos n\theta$, n odd, write θ for $n\theta$: then

$$\sin\theta = 2^{n-1}\sin\frac{\theta}{n}\left(\sin^2 2\alpha - \sin^2\frac{\theta}{n}\right)\left(\sin^2 4\alpha - \sin^2\frac{\theta}{n}\right)\ldots$$

$$= 2^{n-1}\sin^2 2\alpha \sin^2 4\alpha \ldots \sin\frac{\theta}{n}\left(1-\operatorname{cosec}^2 2\alpha \sin^2\frac{\theta}{n}\right)\left(1-\operatorname{cosec}^2 4\alpha \sin^2\frac{\theta}{n}\right)\ldots$$

To determine the value of the constant factor, put $\theta = 0$:

then we have $\sin\theta \div \sin\frac{\theta}{n} = n = 2^{n-1}\sin^2 2\alpha \sin^2 4\alpha \ldots$;

and $\therefore \sin\theta = n\sin\frac{\theta}{n}\left(1-\operatorname{cosec}^2 2\alpha \sin^2\frac{\theta}{n}\right)\left(1-\operatorname{cosec}^2 4\alpha \sin^2\frac{\theta}{n}\right)\ldots$

Put $n = \infty$, then $n\sin\frac{\theta}{n} = \theta$, $\sin\frac{\theta}{n} \div \sin 2\alpha = \frac{\theta}{n} \div \frac{\pi}{n} = \frac{\theta}{\pi}$, &c;

and, consequently, $\sin\theta = \theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\ldots$

So $\cos\theta = 2^{n-1}\sin^2\alpha \sin^2 3\alpha \ldots \cos\frac{\theta}{n}\left(1-\operatorname{cosec}^2\alpha \sin^2\frac{\theta}{n}\ldots\right.$;

put $\theta = 0$; then $1 = 2^{n-1}\sin^2\alpha \sin^2 3\alpha \ldots$,

and $\therefore \cos\theta = \cos\frac{\theta}{n}\left(1-\operatorname{cosec}^2\alpha \sin^2\frac{\theta}{n}\right)\left(1-\operatorname{cosec}^2 3\alpha \sin^2\frac{\theta}{n}\right)\ldots$

$$= \left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\left(1-\frac{4\theta^2}{5^2\pi^2}\right)\ldots, \text{ when } n = \infty.$$

But the above resolutions may be effected more directly as follows.

34. *To resolve* $\sin\theta$ *into a product of quadratic factors in terms of* θ.

Since the values of θ, which make $\sin\theta = 0$, are $0, \pm\pi, \pm 2\pi, \pm 3\pi$, &c., therefore, by the Theory of Equations, we have

$$\sin\theta = a\theta\left(\theta^2-\pi^2\right)\left(\theta^2-2^2\pi^2\right)\left(\theta^2-3^2\pi^2\right)\ldots$$

where a is some constant factor, not known at present : hence

$$\frac{\sin\theta}{\theta} = \pm a\pi^2.2^2\pi^2.3^2\pi^2\ldots\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\ldots$$

Put $\theta = 0$; then $1 = \pm a\pi^2.2^2\pi^2.3^2\pi^2\ldots$, which determines this constant quantity ;

$$\therefore \sin\theta = \theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\ldots$$

35. *To resolve* $\cos\theta$ *into a product of quadratic factors in terms of* θ.

The values of θ which make $\cos\theta=0$ are $\pm\frac{1}{2}\pi$, $\pm\frac{3}{2}\pi$, &c;

$$\therefore \cos\theta=\pm b\left(\theta^2-\frac{\pi^2}{4}\right)\left(\theta^2-\frac{3^2\pi^2}{4}\right)\left(\theta^2-\frac{5^2\pi^2}{4}\right)..$$

$$=\pm b.\frac{\pi^2}{4}.\frac{3^2\pi^2}{4}.\frac{5^2\pi^2}{4}...\left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\left(1-\frac{4\theta^2}{5^2\pi^2}\right)...$$

Put $\theta=0$; then $1=\pm b.\frac{1}{4}\pi^2.\frac{1}{4}(3^2\pi^2).\frac{1}{4}(5^2\pi^2)...$;

$$...\cos\theta=\left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\left(1-\frac{4\theta^2}{5^2\pi^2}\right)...$$

36. In the expression just found for $\sin\theta$, put $\theta=\frac{1}{2}\pi$: then

$$1=\frac{\pi}{2}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{6^2}\right)...=\frac{\pi}{2}.\frac{1.3}{2^2}.\frac{3.5}{4^2}.\frac{5.7}{6^2}...;$$

$$\therefore \frac{\pi}{2}=\frac{2^2}{1.3}.\frac{4^2}{3.5}.\frac{6^2}{5.7}...,$$

which is known as *Wallis's* expression for π.

37. If we expand the expressions in [34], and compare the coefficients of the powers of θ with those in the common formulæ

$$\sin\theta=\theta-\frac{\theta^3}{1.2.3}+\&c,\quad \cos\theta=1-\frac{\theta^2}{1.2}+\&c,$$

we get $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\&c=\frac{\pi^2}{6}$, $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\&c=\frac{\pi^2}{8}$,

with other similar results.

38. *To resolve* e^x+e^{-x} *into quadratic factors.*

In the factorials, [34] and [35], for $\sin\theta$ and $\cos\theta$, use the exponential expressions found in [7]:

then $e^{\theta\sqrt{-1}}-e^{-\theta\sqrt{-1}}=2\sqrt{-1}\,\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)...$

$$e^{\theta\sqrt{-1}}+e^{-\theta\sqrt{-1}}=2\left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)...$$

Write x for $\theta\sqrt{-1}$: then

$$e^x-e^{-x}=2x\left(1+\frac{x^2}{\pi^2}\right)\left(1+\frac{x^2}{2^2\pi^2}\right)...$$

$$e^x+e^{-x}=2\left(1+\frac{4x^2}{\pi^2}\right)\left(1+\frac{4x^2}{3^2\pi^2}\right)...$$

CHAPTER IV.

ON THE SUMMATION OF TRIGONOMETRICAL SERIES.

39. *To find the sums of the sines and cosines of a series of angles in* A. P.

We have $\cos(\theta-\frac{1}{2}\alpha)-\cos(\theta+\frac{1}{2}\alpha)=2\sin\frac{1}{2}\alpha\sin\theta$,

$\cos(\theta+\frac{1}{2}\alpha)-\cos(\theta+\frac{3}{2}\alpha)=2\sin\frac{1}{2}\alpha\sin(\theta+\alpha)$, &c:

hence, by addition,

$$\cos(\theta-\tfrac{1}{2}\alpha)-\cos\{\theta+\tfrac{1}{2}(2n-1)\alpha\}=2\sin\tfrac{1}{2}\alpha\{\sin\theta+\sin(\theta+\alpha)+\&c\};$$

$$\therefore S=\frac{\cos(\theta-\frac{1}{2}\alpha)-\cos\{\theta+\frac{1}{2}(2n-1)\alpha\}}{2\sin\frac{1}{2}\alpha}=\frac{\sin\{\theta+\frac{1}{2}(n-1)\alpha\}\sin\frac{1}{2}n\alpha}{\sin\frac{1}{2}\alpha}.$$

Similarly, we may sum $\cos\theta+\cos(\theta+\alpha)+\&c$; or, if we write $\frac{1}{2}\pi+\theta$ for θ in the above, we get at once

$$S'=\frac{\cos\{\theta+\frac{1}{2}(n-1)\alpha\}\sin\frac{1}{2}n\alpha}{\sin\frac{1}{2}\alpha}.$$

Cor. Putting $\alpha=\theta$ in the above results, we have

$$\sin\theta+\sin2\theta+\&c+\sin n\theta=\frac{\sin\frac{1}{2}(n+1)\theta\sin\frac{1}{2}n\theta}{\sin\frac{1}{2}\theta},$$

$$\cos\theta+\cos2\theta+\&c+\cos n\theta=\frac{\cos\frac{1}{2}(n+1)\theta\sin\frac{1}{2}n\theta}{\sin\frac{1}{2}\theta}.$$

40. The sums of the squares, cubes, &c, of sines and cosines of angles in A. P., may be found by changing the series into one involving the sines and cosines of multiple angles.

Thus, to sum $\cos^3\theta+\cos^3(\theta+\alpha)+\&c.$ to n terms:

here by (75) $\cos^3\theta=\frac{3}{4}\cos\theta+\frac{1}{4}\cos3\theta$;

so that, by [39], we get

$$S=\tfrac{3}{4}\{\cos\theta+\cos(\theta+\alpha)+\&c\}+\tfrac{1}{4}\{\cos3\theta+\cos(3\theta+3\alpha)+\&c\}$$

$$=\frac{3}{4}\frac{\cos\{\theta+\frac{1}{2}(n-1)\alpha\}\sin\frac{1}{2}n\alpha}{\sin\frac{1}{2}\alpha}+\frac{1}{4}\frac{\cos\{3\theta+\frac{1}{2}(n-1)3\alpha\}\sin\frac{3}{2}n\alpha}{\sin\frac{3}{2}\alpha}$$

41. *To sum* $\sin\theta-\sin(\theta+\alpha)+\sin(\theta+2\alpha)-\&c,$

$$\cos\theta-\cos(\theta+\alpha)-\cos(\theta+2\alpha)-\&c.$$

Here $\sin(\theta+\frac{1}{2}\alpha)+\sin(\theta-\frac{1}{2}\alpha)=2\cos\frac{1}{2}\alpha\sin\theta$

$-\sin(\theta+\frac{3}{2}\alpha)-\sin(\theta+\frac{1}{2}\alpha)=-2\cos\frac{1}{2}\alpha\sin(\theta+\alpha)$, &c:

hence $\pm\sin\{\theta+\frac{1}{2}(2n-1)\alpha\}+\sin(\theta-\frac{1}{2}\alpha)=2\cos\frac{1}{2}\alpha\{\sin\theta-\sin(\theta+\alpha)+\&c\}$,

$+$ or $-$ according as n is odd or even;

$$\therefore S=\frac{\sin(\theta-\frac{1}{2}\alpha)+\sin\{\theta+\frac{1}{2}(2n-1)\alpha\}}{2\cos\frac{1}{2}\alpha}$$

$$=\frac{\sin\{\theta+\frac{1}{2}(n-1)\alpha\}\cos\frac{1}{2}n\alpha}{\cos\frac{1}{2}\alpha}, \text{ or } -\frac{\cos\{\theta+\frac{1}{2}(n-1)\alpha\}\sin\frac{1}{2}n\alpha}{\cos\frac{1}{2}\alpha},$$

according as n is *odd* or *even*.

By similar reasoning, or, at once, by writing $\frac{1}{2}\pi+\theta$ for θ in the above, we may get the sum of $\cos\theta-\cos(\theta+\alpha)+$&c; and then, by putting $\alpha=\theta$, we may obtain the sums of

$$\sin\theta-\sin2\theta+\sin3\theta-\&c,\quad \cos\theta-\cos2\theta+\cos3\theta-\&c.$$

42. *To sum* $\sin\theta+x\sin(\theta+\alpha)+x^2\sin(\theta+2\alpha)+$&c,

$$\cos\theta+x\cos(\theta+\alpha)+x^2\cos(\theta+2\alpha)+\&c.$$

Here $x\{\sin(\theta+\alpha)+\sin(\theta-\alpha)\}=2x\sin\theta\cos\alpha,$

$$x^2\{\sin(\theta+2\alpha)+\sin\theta\}\quad=2x^2\sin(\theta+\alpha)\cos\alpha,$$

$$x^3\{\sin(\theta+3\alpha)+\sin(\theta+\alpha)\}=2x^3\sin(\theta+2\alpha)\cos\alpha, \&c,$$

$$\therefore \{S-\sin\theta+x^n\sin(\theta+n\alpha)\}$$

$$+[x\sin(\theta-\alpha)+x^2S-x^{n+1}\sin\{\theta+(n-1)\alpha\}]=2xS\cos\alpha,$$

whence

$$S=\frac{\sin\theta-x\sin(\theta-\alpha)-x^n\sin(\theta+n\alpha)+x^{n+1}\sin\{\theta+(n-1)\alpha\}}{1-2x\cos\alpha+x^2}.$$

By similar reasoning, or, at once, writing $\frac{1}{2}\pi+\theta$ for θ, we get

$$\cos\theta+x\cos(\theta+\alpha)+x^2\cos(\theta+2\alpha)+\&c \text{ to } n \text{ terms}$$

$$=\frac{\cos\theta-x\cos(\theta-\alpha)-x^n\cos(\theta+n\alpha)+x^{n+1}\cos\{\theta+(n-1)\alpha\}}{1-2x\cos\alpha+x^2},$$

These results (by writing $-x$ for x) include the sums of

$$\sin\theta-x\sin(\theta+\alpha)+\&c,\quad \cos\theta-x\cos(\theta+\alpha)+\&c:$$

hence, putting $x=\pm1$, we may get the sums already found, of $\sin\theta\pm\sin(\theta+\alpha)+\sin(\theta+2\alpha)\pm\&c$, $\cos\theta\pm\cos(\theta+\alpha)+\cos(\theta+2\alpha)\pm\&c$; or, again, putting $\alpha=\theta$, we may get the sums of

$$\sin\theta\pm x\sin2\theta+x^2\sin3\theta\pm\&c,\quad \cos\theta\pm x\cos2\theta+x^2\cos3\theta\pm\&c.$$

Cor. If $x<1$, then, when n is very great, x^n and x^{n+1} vanish;

and $\sin\theta\pm x\sin(\theta+\alpha)+\&c$ *ad inf.* $=\dfrac{\sin\theta\mp x\sin(\theta-\alpha)}{1\mp2x\cos\alpha+x^2}$,

$$\cos\theta\pm x\cos(\theta+\alpha)+\&c \text{ } ad\ inf.=\frac{\cos\theta\mp x\cos(\theta-\alpha)}{1\mp2x\cos\alpha+x^2};$$

or, putting $\alpha=\theta$,

$$\sin\theta\pm x\sin2\theta+\&c \text{ } ad\ inf.=\frac{\sin\theta}{1\mp2x\cos\theta+x^2},$$

$$\cos\theta\pm x\cos2\theta+\&c \text{ } ad\ inf.=\frac{\cos\theta\mp1}{1\mp2x\cos\theta+x^2}.$$

43. We shall now illustrate, by a few examples, the use of the results in [13, 14].

Ex. 1. *To sum* $x \sin\theta + x^2 \sin 2\theta + x^3 \sin 3\theta +$ &c *to* n *terms.*

Put $2\sqrt{-1}\sin\theta = z - \frac{1}{z}$; then we have

$$2\sqrt{-1}\,S = (xz - \frac{x}{z}) + (x^2z^2 - \frac{x^2}{z^2}) + \&c = (xz + x^2z^2 + \&c) - (\frac{x}{z} + \frac{x^2}{z^2} + \&c)$$

$$= \frac{(xz)^{n+1} - xz}{xz - 1} - \frac{(xz^{-1})^{n+1} - xz^{-1}}{xz^{-1} - 1}$$

$$= \frac{x^{n+2}(z^n - z^{-n}) - x^{n+1}(z^{n+1} - z^{-(n+1)}) + x(z - z^{-1})}{x^2 - x(z + z^{-1}) + 1};$$

or $$S = \frac{x^{n+2}\sin n\theta - x^{n+1}\sin(n+1)\theta + x\sin\theta}{x^2 - 2x\cos\theta + 1},$$

the same result as we should obtain in [42], by putting $\alpha = \theta$, and multiplying by x.

Ex. 2. *To sum ad. inf.* $(x<1)$

$$x\cos(\theta+\alpha) + \frac{x^2}{1.2}\cos(\theta+2\alpha) + \frac{x^3}{1.2.3}\cos(\theta+3\alpha) + \&c.$$

Let $2\cos\theta = y + y^{-1} = e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}}$, $\quad 2\cos\alpha = z + z^{-1} = e^{\alpha\sqrt{-1}} + e^{-\alpha\sqrt{-1}}$;

then $2\Sigma = x(yz + y^{-1}z^{-1}) + \frac{x^2}{1.2}(yz^2 + y^{-1}z^{-2})$ &c

$$= y(xz + \frac{x^2z^2}{1.2} + \&c) + y^{-1}(xz^{-1} + \frac{x^2z^{-2}}{1.2} + \&c)$$

$$= y(e^{xz} - 1) + y^{-1}(e^{xz^{-1}} - 1)$$

$$= e^{\theta\sqrt{-1}} \times e^{x(\cos\alpha + \sqrt{-1}\sin\alpha)} + e^{-\theta\sqrt{-1}} \times e^{x(\cos\alpha - \sqrt{-1}\sin\alpha)} - (y + y^{-1})$$

$$= e^{x\cos\alpha}\{e^{(\theta + x\sin\alpha)\sqrt{-1}} + e^{-(\theta + x\sin\alpha)\sqrt{-1}}\} - (y + y^{-1})$$

$$= e^{x\cos\alpha} \times 2\cos(\theta + x\sin\alpha) - 2\cos\theta:$$

hence $S = e^{x\cos\alpha}\cos(\theta + x\sin\alpha) - \cos\theta$.

Writing $\frac{1}{2}\pi + \theta$ for θ, we get

$$x\sin(\theta+\alpha) + \frac{x^2}{1.2}\sin(\theta+2\alpha) + \&c = \sin\theta - e^{x\cos\alpha}\sin(\theta + x\sin\alpha).$$

Cor. Putting $\theta = 0$ and $\alpha = \theta$, in these two results, we get

$$x\cos\theta + \frac{x^2}{1.2}\cos 2\theta + \&c = e^{x\cos\theta}\cos(x\sin\theta) - 1,$$

$$x\sin\theta + \frac{x^2}{1.2}\sin 2\theta + \&c = -e^{x\cos\theta}\sin(x\sin\theta).$$

Ex. 3. *To sum ad inf.* $(x<1)$

$$x\cos(\theta+\alpha) + \frac{x(x-1)}{1.2}\cos(\theta+2\alpha) + \frac{x(x-1)(x-2)}{1.2.3}\cos(\theta+3\alpha) + \&c.$$

Reasoning as in Ex. 2, we get

$$2\Sigma = x(yz + y^{-1}z^{-1}) + \frac{x(x-1)}{1.2}(yz^2 + y^{-1}z^{-2}) + \&c.$$

$$= y\{xz + \frac{x(x-1)}{1.2}z^2 + \&c\} + y^{-1}\{xz^{-1} + \frac{x(x-1)}{1.2}z^{-2} + \&c\}$$

$$= y\{(1+z)^x - 1\} + y^{-1}\{(1+z^{-1})^x - 1\}$$

$$= yz^{\frac{1}{2}x}(z^{\frac{1}{2}} + z^{-\frac{1}{2}})^x + y^{-1}z^{-\frac{1}{2}x}(z^{\frac{1}{2}} + z^{-\frac{1}{2}})^x - (y + y^{-1})$$

$$= (yz^{\frac{1}{2}x} + y^{-1}z^{-\frac{1}{2}x})(z^{\frac{1}{2}} + z^{-\frac{1}{2}})^x - (y + y^{-1})$$

$$= 2\cos(\theta + \tfrac{1}{2}x\alpha)(2\cos\tfrac{1}{2}\alpha)^x - 2\cos\theta:$$

$$\therefore \Sigma = 2^x\cos(\theta + \tfrac{1}{2}x\alpha)(\cos\tfrac{1}{2}\alpha)^x - \cos\theta.$$

Writing $\frac{1}{2}\pi + \theta$ for θ, we get

$$x\sin(\theta+\alpha) + \frac{x(x-1)}{1.2}\sin(\theta+2\alpha) + \&c = 2^x\sin(\theta + \tfrac{1}{2}x\alpha)(\cos\tfrac{1}{2}\alpha)^x - \sin\theta.$$

Cor. Putting $\theta = 0$ and $\alpha = \theta$, in these two results, we get

$$x\cos\theta + \frac{x(x-1)}{1.2}\cos 2\theta + \&c = 2^x\cos\tfrac{1}{2}x\theta\,(\cos\tfrac{1}{2}\theta)^x - 1,$$

$$x\sin\theta + \frac{x(x-1)}{1.2}\sin 2\theta + \&c = 2^x\cos\tfrac{1}{2}x\theta\,(\cos\tfrac{1}{2}\theta)^x.$$

44. *To sum* $\sec\theta\sec 2\theta + \sec 2\theta\sec 3\theta + \&c$ *to* n *terms.*

We have $\tan(n+1)\theta - \tan n\theta = \sin\theta\sec n\theta\sec(n+1)\theta$;

hence, putting for n the values 1, 2, 3, &c, successively,

$$\sin\theta\,.\sec\theta\sec 2\theta = \tan 2\theta - \tan\theta,$$

$$\sin\theta\,.\sec 2\theta\sec 3\theta = \tan 3\theta - \tan 2\theta, \&c;$$

$$\therefore \sin\theta\cdot S = \tan(n+1)\theta - \tan\theta, \text{ or } S = \operatorname{cosec}\theta\{\tan(n+1)\theta - \tan\theta\}.$$

45. *To sum* $\operatorname{cosec}\theta + \operatorname{cosec} 2\theta + \operatorname{cosec} 4\theta + \&c$ *to* n *terms.*

We have $\operatorname{cosec}\theta + \cot\theta = \cot\frac{1}{2}\theta$;

hence $\operatorname{cosec}\theta = \cot\frac{1}{2}\theta - \cot\theta,$

$$\operatorname{cosec} 2\theta = \cot\theta - \cot 2\theta, \&c;$$

$$\therefore (S \text{ to } n \text{ terms}) = \cot\tfrac{1}{2}\theta - \cot 2^{n-1}\theta.$$

46. *To sum* $\frac{1}{2}\tan\frac{1}{2}\theta + \frac{1}{4}\tan\frac{1}{4}\theta + \frac{1}{8}\tan\frac{1}{8}\theta + \&c.$

We have $\cot\frac{1}{2}\theta - \tan\frac{1}{2}\theta = 2\cot\theta;$

hence $\frac{1}{2}\tan\frac{1}{2}\theta = \frac{1}{2}\cot\frac{1}{2}\theta - \cot\theta,$

$$\tfrac{1}{4}\tan\tfrac{1}{4}\theta = \tfrac{1}{4}\cot\tfrac{1}{4}\theta - \tfrac{1}{2}\cot\tfrac{1}{2}\theta, \&c;$$

$$\therefore S \text{ (to } n \text{ terms)} = \frac{1}{2^n}\cot\frac{\theta}{2^n} - \cot\theta:$$

and Σ (*ad inf.*) $= \frac{1}{\theta}\left(\frac{\theta}{2^n} \div \tan\frac{\theta}{2^n}\right) - \cot\theta = \frac{1}{\theta} - \cot\theta.$

47. *To sum*

$$(\tan\theta+\cot\theta)+(\tan 2\theta+\cot 2\theta)+(\tan 4\theta+\cot 4\theta)+\&c.$$

As in [46], $\tan\theta+\cot\theta=2\cot\theta-2\cot 2\vartheta$,

$$\tan 2\vartheta+\cot 2\vartheta=2\cot 2\vartheta-2\cot 4\theta, \&c:$$

$\therefore$ S (to n terms) $=2\cot\theta-2\cot 2^n\theta$.

41. It will be sufficient to suggest to the Student, who is acquainted with the processes of the Differential and Integral Calculus, that many series may be summed by differentiating or integrating other series or expressions, found as above.

Thus, [31] $\sin n\theta=2^{n-1}\sin\theta\sin(\theta+\frac{\pi}{n})\sin(\theta+\frac{2\pi}{n})\ldots$;

therefore, taking the logs and differentiating, we have

$$n\cot n\theta=\cot\theta+\cot(\theta+\tfrac{\pi}{n})+\cot(\theta+\tfrac{2\pi}{n})+\&c;$$

and, differentiating again,

$$n^2\operatorname{cosec}^2 n\theta=\operatorname{cosec}^2\theta+\operatorname{cosec}^2(\theta+\tfrac{\pi}{n})=\operatorname{cosec}^2(\theta+\tfrac{2\pi}{n})+\&c.$$

Again, since $\sin\theta=\theta(1-\frac{\theta^2}{\pi^2})(1-\frac{\theta^2}{2^2\pi^2})(1-\frac{\theta^2}{3^2\pi^2})\ldots$,

we get, in the same way,

$$\cot\theta=\frac{1}{\theta}-\frac{2\theta}{\pi^2-\theta^2}-\frac{2\vartheta}{2^2\pi^2-\theta^2}-\&c.$$

So also since, [46], writing 2θ for θ, and multiplying by 2, we have

$$\tan\theta+\tfrac{1}{2}\tan\tfrac{1}{2}\theta+\tfrac{1}{4}\tan\tfrac{1}{4}\theta+\&c \textit{ ad inf.}=\frac{1}{\theta}-2\cot 2\vartheta,$$

we get, by differentiation,

$$\sec^2\theta+\tfrac{1}{4}\sec^2\tfrac{1}{2}\theta+\tfrac{1}{16}\sec^2\tfrac{1}{4}\theta+\&c \textit{ ad. inf.}=-\frac{1}{\theta^2}+4\operatorname{cosec}^2 2\vartheta.$$

49. *Given* $\sin\theta=m\sin(\theta+\alpha)$: *to expand* θ *in a series ascending by powers of* m.

Here $\sin\theta=m(\sin\theta\cos\alpha+\cos\theta\sin\alpha)$, or $\tan\theta=\frac{m\sin\alpha}{1-m\cos\alpha}$:

putting $2\cos\theta=y+y^{-1}$ and $2\cos\alpha=x+x^{-1}$, we get

$$\frac{y-y^{-1}}{y+y^{-1}}=\frac{m(x-x^{-1})}{2-m(x+x^{-1})},\text{ or }\frac{y}{y^{-1}}=y^2=e^{2\theta\sqrt{-1}}=\frac{1-mx^{-1}}{1-mx};$$

taking logs, $2\theta\sqrt{-1}=-(mx^{-1}+\frac{1}{2}m^2x^{-2}+\&c)+(mx+\frac{1}{2}m^2x^2+\&c)$

$$=m(x-x^{-1})+\tfrac{1}{2}m^2(x^2-x^{-2})+\&c,$$

$$\text{or }\theta=m\sin\alpha+\tfrac{1}{2}m^2\sin 2\alpha+\tfrac{1}{3}m^3\sin 3\alpha+\&c.$$

The equation given above may also be put into the form

$$\frac{\sin(\theta+\alpha)+\sin\theta}{\sin(\theta+\alpha)-\sin\theta}=\frac{1+m}{1-m}=\frac{\tan(\theta+\frac{1}{2}\alpha)}{\tan\frac{1}{2}\alpha}, \text{ or } \tan(\theta+\tfrac{1}{2}\alpha)=\frac{1+m}{1-m}\tan\tfrac{1}{2}\alpha,$$

which, of course, would lead to the same result for θ in terms of α.

Since, in any triangle, $\sin A=\frac{a}{b}\sin(A+C)$, in the above write $\frac{a}{b}$ for m and C for α, and let θ represent the circular measure of A'';

then θ or (95) $A\sin 1''=\frac{a}{b}\sin C+\frac{a^2}{2b^2}\sin 2C+\frac{a^3}{3b^3}\sin 3C+$ &c,

a series for expressing one angle of a triangle in terms of another and the two sides containing the latter.

50. *To shew that, in any plane triangle,*

$$\log_e c=\log_e a-\{\frac{b}{a}\cos C+\frac{b^2}{2a^2}\cos 2C+\frac{b^3}{3a^3}\cos 3C+\&c\}.$$

We have $c^2=a^2+b^2-2ab\cos C=a^2+b^2-ab(x+x^{-1})$, if $2\cos C=x+x^{-1}$;

$$\therefore c^2=(a-bx)(a-bx^{-1}), \text{ and } \frac{c^2}{a^2}=(1-\frac{b}{a}x)(1-\frac{b}{a}x^{-1}):$$

$$\therefore 2\log\frac{c}{a}=\log_e(1-\frac{b}{a}x)+\log_e(1-\frac{b}{a}x^{-1})=-\frac{b}{a}(x+x^{-1})-\frac{b^2}{2a^2}(x^2+x^{-2})-\&c;$$

or $$\log_e c=\log_e a-\{\frac{b}{a}\cos C+\frac{b^2}{2a^2}\cos 2C+\frac{b^3}{3a^3}\cos 3C+\&c\}.$$

Cor. Put $a^2+b^2=1$, $2ab=p$, and $C=\alpha$, so that $c^2=1-p\cos\alpha$; then $a\pm b=\sqrt{(1\pm p)}$, or $a=\frac{1}{2}\{\sqrt{(1+p)}+\sqrt{(1-p)}\}$, $b=\frac{1}{2}\{\sqrt{(1+p)}-\sqrt{(1-p)}\}$:

$$\therefore \log_e(1-p\cos\alpha)=2\log_e c=2\{\log_e a-(n\cos\alpha+\tfrac{1}{2}n^2\cos 2\alpha+\&c)\},$$

$$\text{where } n=\frac{b}{a}=\frac{\sqrt{(1+p)}-\sqrt{(1-p)}}{\sqrt{(1+p)}+\sqrt{(1-p)}}=\frac{1-\sqrt{(1-p^2)}}{p}.$$

51. *To expand* $(a^2-2ab\cos\alpha+b^2)^{-m}$ *in a series of the form* $A_0+A_1\cos\alpha+A_2\cos 2\alpha+$ &c.

We have $(a^2-2ab\cos\alpha+b^2)^{-m}$

$=\{a^2-ab(x+x^{-1})+b^2\}^{-m}$, (if $2\cos\alpha=x+x^{-1}$,)

$=(a-bx)^{-m}(a-bx^{-1})^{-m}=a^{-2m}(1-nx)^{-m}(1-nx^{-1})^{-m}$, (if $n=\frac{b}{a}$,)

$=a^{-2m}(c_0+c_1x+c_2x^2+\&c)\times(c_0+c_1x^{-1}+c_2x^{-2}+\&c)$,

(where c_0, c_1, &c are the successive terms of the expansion of $(1-n)^{-m}$,

$=a^{-2m}\{(c_0^2+c_1^2+c_2^2+\&c)+(c_0c_1+c_1c_2+\&c)(x+x^{-1})$

$+(c_0c_2+c_1c_3+\&c)(x^2+x^{-2})+\&c\}$

$=a^{-2m}\{(c_0^2+c_1^2+c_2^2+\&c)+2(c_0c_1+c_1c_2+\&c)\cos\alpha$

$+2(c_0c_2+c_1c_3+\&c)\cos 2\alpha+\&c.\}$

Cor. 1. Hence, as in [50 Cor.], we may expand $(1-p\cos\alpha)^{-m}$.

Cor. 2. If $m=1$, the result of [51] becomes

$$\begin{aligned}(a^2-2ab\cos\alpha+b^2)^{-1} &= a^{-2}(1-nx)^{-1}(1-nx^{-1})^{-1}\\ &= a^{-2}(1+nx+n^2x^2+\&c)\,(1+nx^{-1}+n^2x^{-2}+\&c)\\ &= a^{-2}\{(1+n^2+n^4+\&c)+(n^1+n^3+n^5+\&c)\,2\cos\alpha\\ &\qquad +(n^2+n^4+n^6+\&c)\,2\cos 2\alpha+\&c\},\\ &= a^{-2}\{\frac{1}{1-n^2}(1+2n\cos\alpha+2n^2\cos 2\alpha+\&c)\}:\end{aligned}$$

hence $\dfrac{1}{a^2-2ab\cos\alpha+b^2} = \dfrac{1}{a^2-b^2}\{1+2\dfrac{b}{a}\cos\alpha+2\dfrac{b^2}{a^2}\cos 2\alpha+\&c\}$,

from which, as before, we may get the series for $\dfrac{1}{1-p\cos\alpha}$.

Cor. 3. Since, in any triangle, $c^2=a^2-2ab\cos C+b^2$, we may thus get expressions for c^{-2} and, generally, for c^{-2m} in terms of a, b, C.

52. We have seen that, if r or $+r$ represent a line drawn in any one given direction, then $-r$ will represent a line of the same length, drawn in the opposite direction. This, however, is only a particular case of a far more comprehensive rule of Symbolical Algebra, namely, that, if $+r$ represent a line drawn in any given direction, then a line of the same length, inclined to the former at an angle θ, may be represented by $(\cos\theta+\sqrt{-1}\sin\theta)\,r$.

The truth of the above principle may be best inferred, as in (7), from the correctness of complicated results, obtained by means of it. We shall here, however, illustrate it by some Examples of its application.

And first we may shew that the above formulæ includes the cases already ascertained. For, if $\theta=\pi$, that is, if the second line be drawn in the *opposite* direction to the primary, (to that, namely, expressed by $+r$,) then the formula becomes $(\cos\pi+\sqrt{-1}\sin\pi)\,r$ or $-r$; and, if $\theta=0$ or 2π, that is, if the second line be drawn in the *same* direction with the primary, then the formula becomes $(\cos 0+\sqrt{-1}\sin 0)\,r$ or $(\cos 2\pi+\sqrt{-1}\sin 2\pi)\,r$, that is, it becomes, in each case, $+r$, as it should be.

If $\theta=\frac{1}{2}\pi$, we get $(\cos\frac{1}{2}\pi+\sqrt{-1}\sin\frac{1}{2}\pi)\,r$ or $+r\sqrt{-1}$, as the Symbolical expression for a line, of length r, in the position AD, at right angles to AB; and so, if $\theta=\frac{3}{2}\pi$, (or $-\frac{1}{2}\pi$,) we get $-r\sqrt{-1}$, as the expression for a line, of length r, in the position AD', directly opposite to the former.

Once more, supposing $(\cos\theta+\sqrt{-1}\sin\theta)\ r$ to represent a line, of length r, inclined to the primary at an angle θ, then a line, of length r, inclined to the former at an angle ϕ, would, by the same principle, be expressed by

$$(\cos\phi+\sqrt{-1}\sin\phi)\,(\cos\theta+\sqrt{-1}\sin\theta)r=\{\cos(\phi+\theta)+\sqrt{-1}\sin(\phi+\theta)\}$$

by Demoivre's Theorem—which should be the case, since this last line is inclined to the primary at an angle $(\phi+\theta)$.

Lastly, if the circumference of a circle, beginning from any point B, be divided into n equal parts in P_1, P_2, &c, P_n (or B), then, if the radius $AB=r$, we shall have AP_1, AP_2, &c, expressed (by this principle) by

$$\cos\frac{2\pi}{n}+\sqrt{-1}\sin\frac{2\pi}{n},\ \cos\frac{4\pi}{n}+\sqrt{-1}\sin\frac{4\pi}{n},\ \text{&c.}$$

or [23 Cor.] by ar, a^2r, &c, till we come round again to AP_n (or AB) $=a^nr$, or $+r$, as it should be.

53. With this extension of the mode of expressing symbolically both the magnitude and direction of straight lines, we shall find that the line joining any two points is the symbolical sum of any number of lines that can be drawn (forming the sides of a polygonal figure) from one point to the other.

Ex. 1. In the triangle ABC, right-angled at C, denoting the angle BAC by θ, and the *length* of AB by r, then AB will be expressed symbolically, with reference to AC as the primary or initial line, by

$$(\cos\theta+\sqrt{-1}\sin\theta)r=r\cos\theta+\sqrt{-1}\,r\sin\theta=AC+CB.$$

Or, if we denote the *length* of AC by a, and that of BC by b, then the symbolical expression for AB, namely $(\cos\theta+\sqrt{-1}\sin\theta)r$ becomes

$$\left\{\frac{a}{\sqrt{(a^2+b^2)}}+\sqrt{-1}\,\frac{b}{\sqrt{(a^2+b^2)}}\right\}\sqrt{(a^2+b^2)}=a+b\sqrt{-1}=(\text{as before})\ AC+CB.$$

It appears then that the symbolical expression $a+b\sqrt{-1}$ represents a line, whose length is $\sqrt{(a^2+b^2)}$, inclined to the primary line at an angle $\theta=\tan^{-1}\frac{b}{a}$.

Ex. 2. Completing the parallelogram $ABCD$ in Ex. 1, we have $AD=-\sqrt{-1}\,b$, and also DC (which is drawn from D, parallel to AB and in the *same* direction with it,) $=AB=a+b\sqrt{-1}$; and, consequently, $AD+DC=a=AC$.

Ex. 3. More generally, let ABC be a triangle, and draw BD perpendicular on AC; then by (Ex. 2) we have $AB+BC=(AD+DB)+(BD+DC)=AD+DC=AC$, the symbolical sum of the lines DB and BD being zero.

Ex. 4. Hence $AB+BC+CA=0$, or the symbolical sum of the three sides of a triangle, taken in order, is *zero*:—a result which we may prove *directly*, as follows, using the figure in Ex. 3.

Observing that AB makes with the initial-line the angle A, BC the angle $-C$, (as we find by prolonging BC and AC,) and CA the angle π, we have $AB+BC+CA$

$$=\{\cos A+\sqrt{-1}\sin A\}c+\{\cos(-C)+\sqrt{-1}\sin(-C)\}a+\{\cos\pi+\sqrt{-1}\sin\pi\}b$$

$$=(c\cos A+a\cos C-b)+\sqrt{-1}\,(c\sin A-a\sin C)=0,$$

by (100 Ex. 1) and (105).

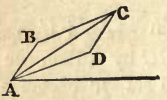
Ex. 5. Since each of the sides in the preceding Example may be broken up into two, and these again, each into two, &c, it is plain that the symbolical sum of the sides of *any* polygon, taken in order, will be zero: or, if the lengths of the sides be a, b, c, &c, making angles α, β, γ, &c, with the primary line, then

$$(\cos\alpha+\sqrt{-1}\sin\alpha)a+(\cos\beta+\sqrt{-1}\sin\beta)b+(\cos\gamma+\sqrt{-1}\sin\gamma)c=\&c=0,$$

whence we get

$$a\cos\alpha+b\cos\beta+\&c=0, \text{ and } a\sin\alpha+b\sin\beta+\&c=0.$$

Ex. 6. In the annexed figure, let the lengths of AB, AD, be r and r', and let the angles made by them with the initial-line be θ, θ', respectively.



Then $AB=DC=(\cos\theta+\sqrt{-1}\sin\theta)r$,

$AD=BC=(\cos\theta'+\sqrt{-1}\sin\theta')r'$;

and $AC=$(Ex. 3) $AB+BC=(r\cos\theta+r'\cos\theta')+\sqrt{-1}\,(r\sin\theta+r'\sin\theta')$, which (Ex. 1) represents a line, whose length is

$$\sqrt{\{(r\cos\theta+r'\cos\theta')^2+(r\sin\theta+r'\sin\theta')^2\}}=\sqrt{\{r^2+r'^2+2rr'\cos(\theta-\theta')\}},$$

inclined to the initial-line at an angle $\phi=\tan^{-1}\dfrac{r\sin\theta+r'\sin\theta'}{r\cos\theta+r'\cos\theta'}$.

For further remarks on this subject the Student is referred to Peacock's Algebra, vol. II. chap. 31.

MISCELLANEOUS EXAMPLES.

1. If $\sin x = \sin a \sin(x+y)$, shew that $\tan x = \dfrac{\sin a \sin y}{1-\sin a \cos y}$.

2. Find the value of $a \cos 2\theta + b \sin 2\theta$, when $\tan\theta = \dfrac{b}{a}$.

3. Shew that $\operatorname{chd}\frac{2}{3}A \operatorname{chd}\frac{1}{3}A = \operatorname{chd}^2 A - \operatorname{chd}^2\frac{1}{3}A$.

4. If $m \operatorname{vers}\theta = n \pm p \operatorname{vers}(\pi-\theta)$, find the value of $\operatorname{vers}\theta$.

5. Shew that the ratios of an English to a Foreign minute and second are $\dfrac{2.5^2}{3.3^2}$ and $\dfrac{2.5^3}{3.3^3}$ respectively.

6. Given the area of a plane triangle in terms of its sides to be
$$\tfrac{1}{4}\sqrt{(2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4)},$$
shew what this expression becomes, when C is a right angle.

7. Shew that the areas of all triangles, described about the same circle, are proportional to their perimeters.

8. The Moon's distance from the Earth being sixty times the Earth's radius, shew that the Earth's radius subtends at the Moon an angle $0°.95$. *See Ex. 177.*

9. The sides of a triangle are 13, 12, and 5 yards: find its area.

10. Given $\log 4.2366 = .6270227$, find the value of $(\frac{1}{3})^{\frac{1.8.7}{1.4}}$.

11. Shew that in any circle the chord of an arc of 108° is equal to the sum of the chords of 36° and 60°.

12. If $\cos A = \cos B \cos C$, shew that
$$\tan\tfrac{1}{2}(A+B)\tan\tfrac{1}{2}(A-B) = \tan^2\tfrac{1}{2}C.$$

173. 13. Express $\cos m\theta \cos n\theta \cos r\theta$ by the sum of a series of cosines of multiples of θ.

14. Shew that, in any triangle,
$$\tan A = \frac{a\sin B}{c-a\cos B} \text{ and } \cot A = \frac{b}{a}\operatorname{cosec} C - \cot C.$$

15. Prove the formula $\cot\dfrac{A}{2} = \sqrt{\dfrac{\sec A+1}{\sec A-1}}$.

16. Shew that
$$\cos(36°+A)\cos(36°-A)+\cos(54°+A)\cos(54°-A) = \cos 2A.$$

17. Determine $\sin A$ from the equation $\tan^2 A + 4\sin^2 A = 1$.

18. A river AC, whose breadth is 200 feet, runs at the foot of a tower BC, which subtends at the edge of the bank an angle BAC of 25° 10′. Find the height of the tower, having given L tan 64° 50′ = 10.3280372, log 5 = .6989700, and log 9.397 = .9729928.

19. Obtain $\sin A$ in terms of $\sin 2A$, when A lies between 180° and 225°.

20. Prove that $1+\cos(2A-2B)\cos 2B = \cos^2 A+\cos^2(A-2B)$.

21. Shew that, if $\sin(A+B)\cos C = 2\cos(B-C)\sin A$, then $\cot A-\cot B = 2\tan C$.

22. If A and B are the two acute angles of a right-angled triangle, shew that $\sin(A-B) = -\cos 2A$, and $\sin 2A+\sin 2B = 2\cos(A-B)$.

23. If α, β, γ, denote the distances from the angular points of a triangle to the points of contact of the inscribed circle, shew that its radius $= \left(\frac{\alpha\beta\gamma}{\alpha+\beta+\gamma}\right)^{\frac{1}{2}}$.

24. Prove that the circular measure of 42° is .73303... .

25. Shew that $\sin A\cos(B-C)-\sin B\cos(A-C) = \sin(A-B)\cos C$.

26. Find the sine, secant, and tangent of 285°.

27. Shew that $\tan^{-1}\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}+\tan^{-1}\sqrt{\frac{3}{2}} = \frac{3}{4}\pi$.

28. Prove the formula $(1+\tan\frac{1}{2}A)^2 = 2\left(\frac{1+\sin A}{1+\cos A}\right)$.

29. Shew that $7\log\frac{16}{15}+5\log\frac{25}{24}+3\log\frac{81}{80} = \log 2$.

30. If A, B, C, are in A. P., shew that
$$\sin A-\sin C : \cos C-\cos A :: \cos B : \sin B.$$

31. If $\tan(\theta+45°)+\tan(\theta-45°) = 2\tan 60°$, determine θ.

32. A man, distant 70 feet from the foot of a tower, observes it to subtend an angle of 60°, his eye being on a level with the ground: find the height of the tower, given log 3 = .4771213, log 7 = .8450980, log 1.21 = .0836586.

33. If $\frac{\cos A-\sin B\sin C}{\cos B\cos C} = \frac{\cos D-\sin E\sin F}{\cos E\cos F}$, shew that
$$\frac{\cos B\cos C}{\cos E\cos F} = \frac{\operatorname{vers} A-\operatorname{vers}(B-C)}{\operatorname{vers} D-\operatorname{vers}(E-F)}.$$

34. Shew that, in any triangle, $\cos A+\cos B = \frac{2(a+b)}{c}\sin^2\frac{1}{2}C$.

35. Taking the Earth's circumference at 25,000 miles, find its diameter.

36. Shew that $\sec(\frac{1}{4}\pi+\theta)\sec(\frac{1}{4}\pi-\theta)=2\sec 2\theta$.

37. One angle of a triangle is 15°, and another is $2\frac{2}{3}$ of the third; shew that the sides opposite to these latter are as $\sqrt{3}:\sqrt{2}$.

38. The perimeter of an equilateral triangle being 27 yards, find the area.

39. CD is the perpendicular from the right angle C of a triangle upon the hypothenuse AB, AC is 108 yds, CB is 144 yards; find CD, BD, and DA.

40. Given $75°$ the sum of two angles and 75^g their difference, express the angles in English and Foreign measure.

41. If the right angle were divided into 100° instead of 90°, what would be the circular measure of 36°?

42. Given $\cos A=.6$, find the values of $\sin A$, $\cot A$, and chord A.

43. Prove that $\sec 2A-\tan 2A=\dfrac{\cos A-\sin A}{\cos A+\sin A}$.

44. If α, β, γ, be any angles, shew that
$\sin\alpha+\sin\beta+\sin\gamma-\sin(\alpha+\beta+\gamma)=4\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha+\gamma)\sin\frac{1}{2}(\beta+\gamma)$.

45. Find the area of a triangle whose sides are $13\frac{1}{2}$, 18, $22\frac{1}{2}$ yds, respectively.

46. Shew that $\sin 30°\ 1'=\cos 1'-\sin 29°\ 59'$.

47. Find the angle whose circular measure is $\frac{5}{6}$.

48. Prove the formula $\dfrac{1+\sin A}{1-\sin A}=\tan(\frac{1}{4}\pi+\frac{1}{2}A)\cot(\frac{1}{4}\pi-\frac{1}{2}A)$.

49. Find the area of a circular sector, where the radius is 9 feet, and the sine $4\frac{1}{2}$ feet.

50. Obtain the acreage of a triangular piece of ground, the sides of which are 380, 420, and 400 yards.

51. Prove the formula $\tan\dfrac{A}{2}=\sqrt{\dfrac{2\sin A-\sin 2A}{2\sin A+\sin 2A}}$.

52. Shew that an inch will subtend an angle of 1″ nearly at a distance of 3 miles.

53. Shew that $\cot^{-1}3+\operatorname{cosec}^{-1}\sqrt{5}=\frac{1}{4}\pi$.

54. If AD, BE, CF, be drawn perpendicular from the angles upon the sides of a triangle, shew that the products of the alternate segments of the sides will be equal.

55. If α, β, γ, be any angles, shew that
$\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma)=4\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha+\gamma)\cos\frac{1}{2}(\beta+\gamma)$.

56. An observer, standing on the bank of a straight canal, observes the angle (α), which the distance of a tree on the other side makes with the canal, and then walks along the bank a distance (a), until the distance of the tree makes the same angle with the canal in the other direction. Find the shortest distance of the tree from the bank.

57. Find the area of a trapezoid, of which the parallel sides are 750 and 1225 links, and the distance between them 1540 links.

58. Prove the formula $\tan(\frac{1}{4}\pi-\frac{1}{2}\theta)+\cot(\frac{1}{4}\pi-\frac{1}{2}\theta)=2\sec\theta$.

59. If $\sin\alpha=m\sin\beta$, and $\tan\alpha=n\tan\beta$, shew that $\cos\alpha=\sqrt{\frac{m^2-1}{n^2-1}}$.

60. Shew that in any triangle $\frac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B}=\frac{a^2+b^2}{a^2+c^2}$.

61. If the right angle were divided into *sixty* degrees instead of *ninety*, what would be the circular measure of 75°?

62. If s, s', be the secants of two arcs, t, t', their tangents, then
$$s^2s'^2-s^2 : s^2s'^2-s'^2 :: t^2t'^2+t'^2 : t^2t'^2+t^2.$$

63. Given $\tan\theta=\frac{2\sqrt{(ab)}}{a-b}$; find the values of $\sin\theta$ and $\operatorname{vers}\theta$.

64. From $\log 2=.3010300$, and $\log 3=.4771213$, find the logs of 72, 45, 1800, .00024.

65. Shew that $\cot\theta-\tan\theta=2\sqrt{\frac{1+\cos 4\theta}{1-\cos 4\theta}}$.

66. Prove that $\sin 48°=\frac{1}{8}\{\sqrt{(10+2\sqrt{5})}+\sqrt{3}(\sqrt{5}-1)\}$.

✓ 67. Shew that the area of a triangle
$$=\tfrac{1}{2}(a^2+b^2+c^2)\frac{c\sin A\sin B}{a\sin A+b\sin B+c\sin C}.$$

68. From two points, A, B, in the diameter produced of a semicircle, tangents AP, BQ, are drawn to the circumference: find the radius CP, given $\angle CAP=\alpha$, $\angle CBQ=\beta$, and $AB=a$.

69. Shew that $\frac{1}{a+b\cos x}=\frac{\sec^2\frac{1}{2}x}{(a+b)+(a-b)\tan^2\frac{1}{2}x}$.

70. If a be a side of a regular polygon of n sides, and R, r, the radii of its circumscribed and inscribed circles, shew that
$$R+r=\tfrac{1}{2}a\cot\frac{\pi}{2n}.$$

71. Shew that the line (CD) drawn from the angle C of a triangle to the base, and bisecting the angle, $=\frac{2ab}{a+b}\cos\frac{1}{2}C$.

72. Find the number whose log is -1.8753145, having given $\log 1.3325 = .1246672$, $\log 1.3326 = .1246998$.

73. Prove that $\sec\theta = \dfrac{3}{\sqrt{\{2+\sqrt{(2+2\cos 4\theta)}\}}}$.

74. If p, q, r, be the perpendiculars upon the sides a, b, c, of a triangle from the opposite angles, shew that

$$\frac{p^2}{qr}+\frac{q^2}{pr}+\frac{r^2}{pq}=\frac{bc}{a^2}+\frac{ac}{b^2}+\frac{ab}{c^2}.$$

75. Shew that $\text{vers}\,(2\pi-\theta) - \text{covers}\,(\frac{3}{2}\pi-\theta) - \text{suvers}\,(\pi-\theta) + \text{suvers}\,(\theta-2\pi) + \text{covers}\,(\theta-\frac{3}{2}\pi) - \text{suvers}\,(\theta-\pi) = 0$.

76. Shew that the area of a triangle $= \frac{1}{4}(a+b+c)^2 \tan\frac{1}{2}A \tan\frac{1}{2}B \tan\frac{1}{2}C$.

77. Find in degrees the angle whose circular measure is .7854.

78. The number of sides of one regular polygon exceeds that of another by 1, and an angle of the first exceeds that of the second by 4°: find the number of sides in each.

79. Given $\tan^3\theta = \dfrac{b}{a}$: find the value of $a\sin\theta + b\cos\theta$, and of $a\sec\theta + b\,\text{cosec}\,\theta$.

80. Obtain the circular measure of $1^{\text{r}}\ 3'\ 75''$, and express the angle in degrees, &c.

81. An arc of 120° in one circle is equal to the whole circumference of another: compare their radii, and find how many degrees of the circumference of the smaller circle are equal in length to the radius of the greater.

82. How many acres are there in a triangular field, whose sides are 13, 14, and 15 chains?

83. Shew that $\text{vers}\,(\frac{m}{m+n}\pi-\theta) + \text{vers}\,(\frac{n}{m+n}\pi+\theta) = \text{vers}\,\pi$.

84. Prove that $\sin 41° + \sin 67° - \sin 31° - \sin 77° = \sin 5°$.

85. The diameter of a circle is divided in the ratio of $1\frac{2}{3} : 2\frac{1}{2}$, and circles are described on the segments as diameters: compare the areas of the three circles.

86. The sum of two angles is m degrees, and the difference n grades: express the greater in degrees, and the less in grades.

87. If, from the angles of a triangle ABC, three lines be drawn to meet in a point O, shew that

$$\sin ABO \cdot \sin BCO \cdot \sin CAO = \sin CBO \cdot \sin ACO \cdot \sin BAO.$$

88. At 225 feet from the foot of a steeple, the elevation was exactly half what it was at 100 feet from it: find the height of the steeple.

89. If $\tan A + \sin A = m$ and $\tan A - \sin A = n$, determine $\cos A$; and shew, by eliminating A, that $(m^2 - n^2)^2 = 16mn$.

90. Solve the equation
$$\sin(x+a) + \cos(x+a) = \sin(x-a) + \cos(x-a).$$

91. Prove that $1 - \frac{\sin^2 A}{1+\cot A} - \frac{\cos^2 A}{1+\tan A} = \frac{1}{2}\sin 2A$.

92. If a foot-square of card-board be divided into four equal squares, by lines bisecting the sides at right angles, how much of it will be left, when the four circles, that can be inscribed in these four squares, are cut out?

93. Find the area of a triangular field, whose sides are 70, 80, 90, yards, respectively.

94. The sides of a triangle being as 3, 4, 5, apply the expression for the tangent of half an angle in terms of the sides to find the largest angle.

95. If C, C', be the two values of the third angle in the ambig. case, when a, b, A, are given, and $b > a$, then $\tan A = \cot \frac{1}{2}(C + C')$.

96. Find the length of the arc, which subtends an angle 6° 12′ 36″ in a circle whose radius is 100 yards.

97. A mill-sail is 7 yds in length, and goes round uniformly 10 times in a minute. At what rate per hour does its extremity move?

98. Prove the formula $\tan(A+30°)\tan(A-30°) = \frac{1-2\cos 2A}{1+2\cos 2A}$.

99. Find the number of degrees in the arcs AE, $BCDF$, in the figure of Euc. IV. 16: and, if BC be joined, find the angles of the triangle ABC.

100. How many cubic inches of iron will be required to form a garden roller, which is half-an-inch thick, with an outer circumference of $5\frac{1}{2}$ feet, and a width of $3\frac{1}{2}$ feet? ($\pi = \frac{22}{7}$).

101. Shew that
$$2\cos A = \sqrt{\{2 + \sqrt{(2 + \sqrt{(2 + \ldots\ldots + \sqrt{(2 + 2\cos 2^n A)}}},$$
$$2\sin A = \sqrt{\{2 - \sqrt{(2 + \sqrt{(2 + \ldots\ldots + \sqrt{(2 + 2\cos 2^n A)}\}}}.$$

102. Eliminate ϕ between the two equations
$$x = 3\cos\phi + \cos 3\phi, \quad y = 3\sin\phi - \sin 3\phi.$$

103. Find the value of z in the equation $2\sqrt{(z^2 - z^4)} = \sin A$.

104. If $\tan\theta = \frac{b\sin\phi}{a + b\cos\phi}$, shew that $\tan(\phi - \theta) = \frac{a\sin\phi}{b + a\cos\phi}$.

105. In any triangle, shew that $a^2 \sin 2B + b^2 \sin 2A = 2ab \sin C$.

106. Prove that $\sin \alpha \cos(\beta-\gamma) - \sin\beta \cos(\alpha-\gamma) = \sin(\alpha-\beta)\cos\gamma$.

107. If r be the radius of the inscribed circle of a triangle, shew that the product of the three perpendiculars from the angles on the sides will be $\dfrac{(a+b+c)^3}{abc} r^3$.

108. Given Log-sin 59° 37′ 40″ = 9.9358894, and Diff. for 10″ = 124, find A' from the equation Log-sin A' = 9.9358921, and find also Log-sin A'', where A'' = 59° 37′ 45″.34.

109. Prove that $\sin^{-1}\sqrt{\dfrac{x}{a+x}} = \tan^{-1}\sqrt{\dfrac{x}{a}} = \frac{1}{2}\cos^{-1}\dfrac{a-x}{a+x}$.

110. Find the sine, cosine, and tangent of $(m+\frac{1}{2})\pi+\theta$.

111. Find the value of θ when $\sin(\theta+\frac{1}{4}\pi)\sin(\theta-\frac{1}{4}\pi) = \sin^2\frac{1}{8}\pi$.

112. Given $\log 1\frac{1}{4} = .0969100$ and $\log .1 = \bar{1}.0457575$, find the logs of $2\frac{1}{2}$, $2\frac{1}{4}$, and $.2\dot{}$.

113. AB, AC, both measured in the same direction from A, are the sides of a regular octagon and heptagon, inscribed in the same circle: find the number of degrees in the angle BAC, and find also the interior angle of a regular figure, one of whose sides shall be BC.

114. Prove that $\tan(\alpha+\beta) = \dfrac{\sin^2\alpha - \sin^2\beta}{\sin\alpha\cos\alpha - \sin\beta\cos\beta}$.

115. From the arc AB of a circle, whose centre is O, AC is cut off equal to the sine of AB: shew that the sector BOC is equal to the segment ACB.

116. If p be the perpendicular from the angle C of a triangle upon the opposite side, shew that

$$\frac{a\sin A + b\sin B + c\sin C}{bc\cos A + ac\cos B + ab\cos C} = \frac{2p}{ab}.$$

117. If R, r, be the radii of the circles described about and in the triangle ABC, shew that the area of the triangle

$$= Rr(\sin A + \sin B + \sin C).$$

118. If $\cos(\alpha+\beta)\sin(\gamma+\delta) = \cos(\alpha-\beta)\sin(\gamma-\delta)$, shew that

$$\tan\delta = \tan\alpha\tan\beta\tan\gamma.$$

119. In any triangle, shew that $\dfrac{\sin A}{\sin B} = \dfrac{\cos A\cos C + \cos B}{\cos B\cos C + \cos A}$.

120. Apply the last result to shew that $\sin A : \sin B :: a : b$, assuming the expression for the cosine of an angle of a triangle in terms of the sides.

121. Shew that, in any plane triangle, $c=(a-b)\sec\phi$, where ϕ is given by the formula $\tan\phi=\frac{2\sqrt{(ab)}}{a-b}\sin\frac{1}{2}A$. What objection would there be to using this formula, if $a=b$ nearly?

122. Find the value of A in the equation
$$\tan(A+45°)+\tan(A-45°)=2\tan 60°.$$

123. Prove (by means of the Binomial Theorem) that
$$(1+2\cos^2\theta+3\cos^4\theta+\&c)\,(1-2\tan^2\theta+3\tan^4\theta-\&c)=\cot^4\theta.$$

124. Given $\cos nA+\cos(n-2)A=\cos A$: find A in general terms.

125. Given that $\sin(\beta+\theta)+\sin\alpha\,(\sin^2\beta-\sin^2\alpha)+\sin\{2\alpha-(\beta-\theta)\}$ $=(\cos^2\beta-\cos^2\alpha)\sin(\alpha+2\theta)$, shew that $\sec\theta=\sin(\alpha+\beta)\tan(\alpha-\beta)$.

126. Given $\log 2=.3010300$, and $\log 3=.4771213$, find the logs of $\sqrt{\frac{1}{2}}$, $\sqrt[3]{\frac{1}{3}}$, and $(3\frac{1}{5})^{-\frac{1}{3}}$.

127. If $\alpha+\beta+\gamma=\pi$, shew that
$$\tan\left(\frac{2^n-1}{2^n}\cdot\frac{\pi}{3}+\frac{\alpha}{2^n}\right)+\tan\left(\frac{2^n-1}{2^n}\cdot\frac{\pi}{3}+\frac{\beta}{2^n}\right)+\tan\left(\frac{2^n-1}{2^n}\cdot\frac{\pi}{3}+\frac{\gamma}{2^n}\right)$$
$=$ the product of the same three tangents.

128. Three objects A, B, C, form a triangle, whose angles are as the numbers 1, 4, 1. B is inaccessible from A or C; but an observer, walking from A towards C, measures a base AD (a feet) and observes the angle BDC; he then advances b feet farther to E, and observes the angle BEC to be the supplement of the former angle. Find the distance of B from A: and shew that $AB=DE\tan BDE$.

129. In any plane triangle, $c=a\cos B\pm\sqrt{(b^2-a^2\sin^2B)}$.

130. Given $\log 2=.30103$, find the logs of 5, .016, $\frac{1}{2}$, 6.25, $1\frac{3}{5}$, $15\frac{5}{8}$.

131. Given $\cot^{-1}(x-1)-\cot^{-1}(x+1)=\frac{1}{12}\pi$, determine x.

132. Prove that $\dfrac{\tan A+\sec A}{\cot A+\operatorname{cosec} A}=\tan(45°+\frac{1}{2}A)\tan\frac{1}{2}A$.

133. If $\dfrac{m\tan(\alpha-x)}{\cos^2x}=\dfrac{n\tan x}{\cos^2(\alpha-x)}$, then $x=\frac{1}{2}\{\alpha-\tan^{-1}(\frac{n-m}{n+m}\tan\alpha)\}$.

134. The area of a regular polygon, inscribed in a circle: that of the corresponding circumscribed polygon :: 3 : 4; find the number of sides.

135. Eliminate θ between the equations
$(a+b)\tan(\theta-\phi)=(a-b)\tan(\theta+\phi)$ and $a\cos 2\phi+b\cos 2\theta=c$.

136. One angle of a triangle is 18° and another 54°, and the least side is 96 feet: find the greatest side.

137. If $\tan\theta = \sec\alpha - \tan\alpha$, find θ in terms of α.

138. The excess of the sine above the versed-sine, in angles less than 90°, is greatest when the angle is 45°. Prove this, and find the *maximum* value of the excess.

139. Find the changes in sign and magnitude of the quantity $\sin\theta + \cos\theta$, while θ changes from 0° to 360°.

140. Find, in a form adapted to logarithmic computation, the area of a trapezium, whose two *parallel* sides are a and b, and c the distance between them.

141. Determine $\cos\phi$ from the equation $\tan\frac{1}{2}\phi = \operatorname{cosec}\phi - \sin\phi$.

142. The angles of an inscribed quadrilateral, taken in order, when multiplied by 1, 2, 2, 4, respectively, are in A. P.: find them.

143. Prove that $1 \pm \sin\theta = 2\sin^2(\frac{1}{4}\pi \pm \frac{1}{2}\theta)$.

144. Determine A from the equation $1 + 2\sin 4A = 4\sin 3A\cos A$.

145. Shew that the equation $\sin\theta\sin(2\alpha+\theta) + n\cos^2\alpha = 0$ cannot hold, unless $\sec\alpha > \sqrt{(1+n)}$.

146. If chord $\theta = \cos\theta$, shew that $\theta = 2\cos^{-1}\sqrt{\cos\frac{1}{8}\pi}$.

147. A person, from the top of a tower, observes the angles of depression (α, β) of two objects, whose distance (a) from each other he knows, and which lie just in a line with the tower. Determine its height.

148. Find α from the equation $\tan\frac{1}{2}\alpha = \operatorname{cosec}\alpha$.

149. Shew that

$$\{\tan(A+45°)+\tan(A-45°)\} \div \{\cot(45°+A)+\cot(45°-A)\} = \sin 2A.$$

150. Eliminate θ between the equations

$$m = \operatorname{cosec}\theta - \sin\theta, \quad n = \sec\theta - \cos\theta.$$

151. If $(x-a)\cos\delta + c\sin(l-\delta) = 0$, $(y-b)\cos\delta + c\cos(l-\delta) = 0$, prove that $(x-a)\sin l + (y-b)\cos l + c = 0$.

152. Given $\beta = 4\alpha^2$, where α is the circular measure of $3'\ 47''$, shew that the value of β is $1''$, very nearly.

153. Solve the equations (i) $\cos A = \cos 2A$, (ii) $\tan 2A = 3\tan A$.

154. If a triangle ABC be formed, by joining the centres of three circles which touch one another, and if s be the semi-sum of its sides a, b, c, then the radii of the circles, whose centres are A, B, C, respectively, will be $s-a, s-b, s-c$.

155. At the distance of 50 miles from a tower, its top just appeared in the horizon; determine its height, having given the Earth's diameter to be 7964 miles.

156. Shew that $1+\sin^2 A\cot^2 A-2\sin A\cot A+\cos^2 A\tan^2 A=\mathrm{chd}^2 A$.

157. Given $1-\sin\alpha=2\sin^2(x-\frac{1}{2}\alpha)$, find x.

158. If the side of a regular inscribed polygon be an m^{th} part of the radius, shew that the secant of the angle subtended by it at the centre will be $\dfrac{2m^2}{2m^2-1}$.

159. Given tab-sin 18° 1′ = 3092.936, find tab-sin 18° 0′ 23″.

160. Shew that $\cos^2\theta+\dfrac{\sin\theta\,(1+\sin\theta)\,(1-\tan\frac{1}{2}\theta)}{1+\cot\frac{1}{2}\theta}=\cos\theta$.

161. Find the area of a circle, traced by a pair of compasses, the length of whose legs is 6 inches, and the angle between them 120°.

162. Solve the equation $\tan A\,(\tan 2A+\cot A)=2$.

163. Prove that the square of the side of an inscribed pentagon equals the sum of the squares of the sides of a hexagon and decagon, inscribed in the same circle.

164. A general, having got possession of a town, ascends a tower, and sees a river, whose breadth he wishes to ascertain. He first determines the height of the tower above the horizontal plane to be 210 feet, and then observes the angles of depression of the two banks of the river, where it crosses the plain directly before him, to be 23° and 25°. Shew how he may find the width of the stream, having given

L sin 2° = 8.5428192, L sin 23° = 9.5918780, L sin 25° = 9.6259483, log 3 = .4771213, log 7 = .8450980, log 4.43 = .6472122.

165. Determine ϕ when $\tan\phi+2\cot 2\phi=\sin\phi\,(1+\tan\phi\tan\frac{1}{2}\phi)$.

166. Prove that $\cos^6\alpha-\sin^6\alpha=\cos 2\alpha\,(1-\frac{1}{4}\sin^2 2\alpha)$.

167. In any triangle, shew that

$$a\cos A+b\cos B+c\cos C=4R\sin A\sin B\sin C,$$

where R is the radius of the circumscribing circle.

168. The sides a, b, c, of a triangle are as the numbers 4, 5, 6: find the angle B, having given log 2 = .3010300, L cos 27° 53′ = 9.9464040, L cos 27° 54′ = 9.9463371.

169. Obtain the numerical value of sec 67° 30′.

170. Shew that $\dfrac{\sin 2\alpha+\cos 2\alpha}{(\cos\alpha-\sin\alpha)-(\cos 3\alpha-\sin 3\alpha)}=\frac{1}{2}\operatorname{cosec}\alpha$.

171. Put the expression $a\cos\theta+b\cos(\theta+\alpha)$ into the form $k\cos(\theta+\beta)$, determining the values of k and β.

172. Two observers, a mile apart from each other, on the same side of a balloon, and in the same vertical plane with it, find its angles of elevation at the same moment to be 30° and 75°. Find the height of the balloon.

173. Shew that

$$\cot\theta+\cot 2\theta+\cot 4\theta=\operatorname{cosec} 4\theta\,(2+2\cos 2\theta+3\cos 4\theta).$$

174. Prove that the following rule for finding the distance of the horizon at sea is very nearly correct: take the number of feet in the height of the station above the level of the sea, and increase it by half that number; then the square root of this quantity will give the distance of the horizon in miles.

175. The sides of a triangle are to each other as 9 : 7, and the included angle is 64° 12′: determine the other angles, having given $\log 2=.3010300$, L tan 57° 54′ = 10 2025255, L tan 11° 16′ = 9.2993216, L tan 11° 17′ = 9.2999804.

176. Express $\sin^3\theta\cos^2\theta$ by a series of sines of multiples of θ.

177. Find the distance from the eye, at which a circular disc of six inches diameter must be placed, so as just to cover the Moon, whose apparent diameter is 0°.95. mistake. see Ex. 8.

178. Transform the equation $\surd(2\sec A)=\frac{1}{2}\sqrt[3]{\dfrac{\cos^2 B}{\sin C}}$ into the equation between the tabular logarithms of the quantities.

179. Prove that

$$\operatorname{cosec}^2\theta+\operatorname{cosec}^2\phi-2\operatorname{cosec}\theta\operatorname{cosec}\phi\cos(\theta-\phi)=(\cot\theta\sim\cot\phi)^2.$$

180. In making the circuit of a field, I walked 30 yards due S, then 60 yards due E, then 110 yards due N, and then straight to the place of first starting. Find the circumference and area of the field.

181. If lines be drawn from the angles of a triangle ABC to the centre of the inscribed circle, cutting its circumference in D, E, F, shew that the angles of the triangle DEF are, respectively, $\frac{1}{4}(\pi+A)$, $\frac{1}{4}(\pi+B)$, $\frac{1}{4}(\pi+C)$.

182. Eliminate θ between the equations

$$\cos(\phi-\theta+\alpha)\cos(\theta-\alpha)=\cos(\phi-\theta-\alpha)\cos(\theta+\alpha)=c.$$

183. Write down the values of $\sec\frac{1}{3}m\pi$, m being an integer.

184. Express $\tan A$ in terms of $\tan 2A$, and exhibit by a geometrical figure the two values of $\tan A$.

185. Solve the equation $8^x.\,125^{2-x}=2^{4x+3}.\,5^x$, given $\log 2=.30103$.

186. The hypothenuse AB of a right-angled triangle is trisected in D, E: prove that the sum of the squares of the sides of the triangle CDE is $\frac{2}{3}c^2$.

187. If the fraction $\frac{a\cos(\theta+\alpha)+b\sin\theta}{a'\sin(\theta+\alpha)+b'\cos\theta}$ remain the same for all values of θ, shew that $aa'-bb'=(a'b-ab')\sin\alpha$.

188. If the tangents of the semiangles of a triangle be in A. P., shew that the cosines of the angles will be also in A. P.

189. Given $\frac{\sin\theta}{\theta}=\frac{2165}{2166}$, shew that θ contains 3° nearly.

190. Given L sin 17° = 9.4659353, L sin 17° 1′ = 9.4663483, find L sin 17° 12″, and the angle whose L sin is 9.4659894.

191. If $f(x)=a\cos x\theta+b\sin x\theta$, shew that
$$f(x+2)-2\cos\theta\,.\,f(x+1)+f(x)=0.$$

192. If $\sin\phi=m\tan\theta$ and $\sin\theta=n\tan\phi$, find $\cos\theta$ in terms of m and n.

193. Prove that
$$\frac{\sin\alpha}{\sin(\alpha-\beta)\sin(\alpha-\gamma)}+\frac{\sin\beta}{\sin(\beta-\gamma)\sin(\beta-\alpha)}+\frac{\sin\gamma}{\sin(\gamma-\alpha)\sin(\gamma-\beta)}=0.$$

194. Find the radius of a circle, when six inches is the length of the versed-sine of an arc whose chord is a yard.

195. The side BC of a triangle is bisected in D: shew that
$$\cot BAD\sim\cot CAD=\cot B\sim\cot C.$$

196. Find A in the equation L cos A = 9.9694579,
given L cos 21° 15′ = 9.9694196, and Diff. for 1′ = 491:
find also L cosec 68° 45′ 24″.

197. If $\frac{1}{2}(\tan\theta+\cot\theta)=\frac{m^2+n^2}{m^2-n^2}$, determine $\tan\theta$ and $\cos 2\theta$.

198. Prove that tan A, tan B, tan C, are in H. P., if
$$\cos(A-C)\cos B=\cos(A-B+C).$$

199. Find the general value of A, when $\cos A+\sin A=\sqrt{2}$.

200. One angle of a triangle is 32° 15′, the sides including another are 468 and 320 yards; find the latter angle, given
log 2 = .3010300, log 3 = .4771213, log 13 = 1.1139434,
L cosec 32° 15′ = 10.2727724, L sin 21° 23′ = 9.5621316,
L sin 51° 18′ = 9.8923236.

201. Determine θ from the equation $\cos\theta+\cos 2\theta+\cos 3\theta=0$.

202. Shew that any triangle may be divided into four others, three of which shall be isosceles.

203. If A', B', C', be the angles subtended by the sides of a triangle at the centre of its inscribed circle, shew that

$$4 \sin A' \sin B' \sin C' = \sin A + \sin B + \sin C.$$

204. At 120 feet distance from the foot of a steeple, the angle of elevation of the summit was found to be 60° 14″. Find its height having given $\tan 60° 1' = 1.7332149$.

205. Find the general value of A when $\cos 2A = \sin A$.

206. Shew that $A + \tan^{-1}(\cot 2A) = \tan^{-1}(\cot A)$.

207. If the sides of a triangle ABC be divided in D, E, F, so that $\frac{AF}{BF} = \frac{BD}{CD} = \frac{CE}{AE} = \frac{1}{n-1}$, prove that the area of the triangle formed by joining the points of section : area of original triangle $:: n^2 - 3n + 3 : n^2$.

208. At the ends of a horizontal base ($AB = a$), whose bearing (ϕ) from the North is known, the angles (α, β) are observed, subtended by the lines (BC, AC) which join the two ends of the base with the foot (C) of a certain pillar. Given also (θ) the elevation of the pillar at the end A, find its height and bearing from the North.

209. From the top of the Peak of Teneriffe, the dip of the horizon is found to be 1° 58′. If the Earth's radius be taken as 4000 miles, find the height of the mountain, given sec 1° 58′ = 1.0005894.

210. Given $a = 55$, $b = 40$, $A = 54° 10'$, L $\sin A = 9.9088727$, L $\sin 36° 7' = 9.7704332$, L $\sin 36° 8' = 9.7706063$: find B.

211. If l be the length of the line, drawn from the vertex C of a triangle to bisect the base ($2c$), shew that the area of the triangle $= \frac{1}{2}(l^2 - c^2) \tan C$.

212. The tangents of the angles of a triangle are as the numbers 1, 2, 3: find them, and shew that if p, q, r, be the perpendiculars upon the opposite sides from the angles A, B, C, then $pqr = \frac{3}{2}abc$.

213. Given $\cos nA + \cos(n-2)A = \cos A$, determine A.

214. Shew that $\cos^2\varsigma + \cos^2(\varsigma - \theta) + \cos^2(\varsigma - \phi) + \cos^2(\varsigma - \psi)$
$= 2 + 2\cos\theta\cos\phi\cos\psi$, if $2\varsigma = \theta + \phi + \psi$.

215. Find the tangents of 36° and $37\frac{1}{2}°$.

216. Adapt $a\cos\theta \pm b\sin\theta$ to logarithmic computation.

217. The sides a, b, c, of a triangle are as the numbers 4, 5, 6: shew that $\cos B = \cos^2 A$, and, having given L $\cos 55° 46' = 9.7501723$, L $\cos 55° 47' = 9.7499866$, find the angle B.

218. If c be the chord of a quadrantal arc, a its radius, and a' the radius of the circle inscribed in the quadrant, shew that $a+a'=c$.

219. An equilateral triangle and a regular hexagon have the same perimeter: shew that the areas of their inscribed circles are as 4 : 3.

220. Prove that the area of a triangle
$$=\tfrac{1}{4}(a^2+b^2-c^2)\cot\tfrac{1}{2}(A+B-C).$$

221. Given $\cot A \sin C=\cot a \sin b-\cos b \cos C$, find C in a form adapted to logarithmic computation.

222. Given $\sin 4\theta+4\sin 3\theta\cos\theta=0$; find $\tan\theta$.

223. If the hypothenuse (AB) of a right-angled triangle be divided in D by a line which bisects the right angle, then
$$AD : BD :: 1-\tan\tfrac{1}{2}(A-B) : 1+\tan\tfrac{1}{2}(A-B).$$

224. Solve the equation $2\cos 2A=2\sin A+1$.

225. From a station B, at the base of a mountain, its summit A is seen at an elevation of 60°; and, after walking one mile towards the summit, up a plane making an angle of 30° with the horizon, to another station C, the angle BCA is observed to be 135°. Find, in feet, the height of the mountain above the horizontal plane at B.

226. If $\sec a \sec\theta+\tan a \tan\theta=\sec\beta$,
then $\sec a \tan\theta+\tan a \sec\theta=\tan\beta$.

227. Shew that $\sin A+\cos A+\sin B+\cos B$
$$=2\sqrt{2}\cos\{45°-\tfrac{1}{2}(A+B)\}\cos\tfrac{1}{2}(A-B).$$

228. Prove that $\cos^4\frac{1}{8}\pi+\cos^4\frac{3}{8}\pi+\cos^4\frac{5}{8}\pi+\cos^4\frac{7}{8}\pi=\frac{3}{2}$.

229. The elevation of a steeple, standing upon a horizontal plane, is observed, and, at a station a feet nearer to it, its elevation is found to be the *complement* of the former. On advancing in the same direction b feet nearer still, the elevation is found to be *double* of the first. Shew that the height of the steeple is $\sqrt{\{(a+b)^2-\frac{1}{4}a^2\}}$.

230. Shew that $\sqrt{\text{vers}\,\theta}=\dfrac{\sin\theta}{\sin(45°-\frac{1}{2}\theta)+\cos(45°-\frac{1}{2}\theta)}$.

231. If $A+B+C=180°$, shew that
$$\cos\tfrac{1}{2}A+\cos\tfrac{1}{2}B+\cos\tfrac{1}{2}C=4\cos\tfrac{1}{4}(A+B)\cos\tfrac{1}{4}(B+C)\cos\tfrac{1}{4}(C+A).$$

232. Find x, when $\tan^{-1}\frac{1}{4}+2\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{6}+\cot^{-1}x=\frac{1}{2}\pi$.

233. A, B, C, are three stations. From B a base BD (a) is measured along BA: also the angles CBD, BDC, BCA, (θ, ϕ, ψ) are observed, as well as the depression (α) of A and the elevation (β) of B with respect to C. Find the height of B above the horizontal plane through A.

234. If, in any plane triangle, lines (l, m, n) be drawn bisecting the angles A, B, C, respectively, then

$$l^2\left(\frac{1}{c}-\frac{1}{b}\right)+m^2\left(\frac{1}{a}-\frac{1}{c}\right)+n^2\left(\frac{1}{b}-\frac{1}{a}\right)=0.$$

235. Shew that $\sqrt{(1+\sin\theta)}=1+2\sin\tfrac{1}{4}\theta\sqrt{(1-\sin\tfrac{1}{2}\theta)}$.

236. Find x, when $\cos(x+\frac{1}{3})\alpha+\cos(x+\frac{1}{2})\alpha=\sin\alpha$.

237. If in any triangle the angles A, B, C, be in A. P., shew that $\cot\frac{1}{2}A$, $\cot\frac{1}{2}B$, $\cot\frac{1}{2}C$, will be also in A. P.

238. Shew that $\cos^{-1}\sqrt{\frac{2}{3}}-\cos^{-1}\frac{\sqrt{6}+1}{2\sqrt{3}}=\frac{\pi}{6}$.

239. Given the perimeter and the three angles of a triangle, determine one of its sides.

240. If $\tan\theta=\cos\alpha\tan\phi$, shew that $\tan(\phi-\theta)=\frac{\tan^2\frac{1}{2}\alpha\sin 2\phi}{1+\tan^2\frac{1}{2}\alpha\cos 2\phi}$.

241. A man ascends a mountain by a path, which is the shortest distance between the base and vertex. The inclination of his path to the horizon is at first α; but afterwards it increases suddenly to β, and then continues unaltered. At the top he finds by the barometer that he has ascended m *feet* in vertical height, and observes the angle of depression (γ) of his starting-place. Find the distance travelled by him in the ascent.

242. Shew that in any triangle the length of the line from C, bisecting the side c, is $\frac{1}{2}\sqrt{\{2(a^2+b^2)-c^2\}}$.

243. Find x in the equation $\sec x=2\sin^2 x+\cos x$.

244. Prove that $\frac{1+(\operatorname{cosec}\alpha\tan x)^2}{1+(\operatorname{cosec}\beta\tan x)^2}=\frac{1+(\cot\alpha\sin x)^2}{1+(\cot\beta\sin x)^2}$.

245. Shew that $\cos^{-1}\frac{1-m^2}{1+m^2}-\cos^{-1}\frac{1-n^2}{1+n^2}=\tan^{-1}\frac{2(m-n)(1+mn)}{(1+mn)^2-(m-n)^2}$.

246. Given α, β, γ, the distances from the angles of the point at which the sides of a plane triangle subtend equal angles: find the side c, and the area of the triangle.

247. Shew that $\tan^{-1}\frac{x\cos\phi}{1-\sin\phi}-\tan^{-1}\frac{x-\sin\phi}{\cos\phi}=\frac{\pi}{4}$.

248. Find the general value of A in the following equations:
(i) $\sin A=\pm\frac{1}{2}\sqrt{2}$; (ii) $\cos A=\pm\frac{1}{2}$; (iii) $\cos A+\sin A=\sqrt{2}$.

249. In the figure of *Euc.* IV. 10, shew that the areas of the two circles are as $5+\sqrt{5}:2$.

250. Shew that the equation $\cos(a+x)=\cos a\sin x+\sin b$ may be reduced to the two equations
$\cos(\phi+x)=\sin b\sec a\cos\phi$, $\tan\phi=\sin(45°+a)\sec 45°\sec a$.

251. Solve the equation $\tan\theta+\cot\theta=4$; and express the values of θ by a single formula.

252. Prove that $\cot^{-1}n-\cot^{-1}(n+1)=\cot^{-1}(1+n+n^2)$; and thence sum $\cot^{-1}(1+1+1^2)+\cot^{-1}(1+2+2^2)+$&c. to n terms.

253. Straight lines m, n, are drawn from the acute angles (A, B) of a right-angled triangle to the bisections (D, E) of the opposite sides. Express the tangents of these angles in terms of m and n.

254. Prove that $\dfrac{\sin 3A+\cos 3A}{\sin 3A-\cos 3A}=\dfrac{2\sin 2A+1}{2\sin 2A-1}\times\tan(45°-A)$.

255. The angles of a triangle form a G. P., whose ratio is $\frac{1}{2}$: shew that the greatest side$=2$(perimeter) $\sin 12°\ 51'\ 25\frac{5}{7}''$.

256. If $\sin^2A+\sin^2B=\sin^2C$, is the triangle right-angled or not?

257. If in the three edges, which meet in one angle of a cube, three points A, B, C, be taken at distances a, b, c, from the angles respectively, then the area of the triangle ABC
$$=\tfrac{1}{2}\sqrt{(a^2b^2+a^2c^2+b^2c^2)}.$$

258. Shew from *Euc.* XI. 21, that there can be only five regular solids, and that in these the faces will be either equilateral triangles, squares, or regular pentagons.

259. In a plane triangle, if $a+b=m$, shew that $a=m\cos^2\phi$, $b=m\sin^2\phi$, where ϕ may be found from the equation
$$m\sin 2\phi=\sqrt{\{(m+c)(m-c)\}}\sec\tfrac{1}{2}C.$$

260. Find the value of $\cos^{-1}\frac{1}{2}(-1)^m$, m being any integer; and express the different results by one single formula.

261. If p, q, r, be the lines (AD, BE, CF) drawn from the angles of a triangle to bisect the sides a, b, c, respectively, express a side (c) in terms of p, q, r.

262. If $\tan A=\sin 2A$, find A and chord A.

263. Employ a subsidiary angle to adapt $\sqrt{(a+b)}+\sqrt{(a-b)}$ to logarithmic computation.

264. Shew that the length of the line CD, which is drawn to the side AB of a triangle, so as to bisect the angle C, is

$$\frac{2ab}{a+b}\cos\frac{C}{2}.$$

265. A, B, C, are three points in a horizontal plane, at known distances a, b, c, from each other, at each of which the height of a tower, whose foot is in the same plane, is seen under the same angle (α). Determine it.

266. Shew that the equation $\sin\theta+\sin\phi=\sin(\theta+\phi)$ can only hold when one of the quantities, θ, ϕ, or $\theta+\phi$, is a multiple of 360°, (including *zero*).

267. In a right-angled triangle, lines CD, CE, are drawn from the right-angle C, making towards the same parts angles α, β, with the hypothenuse: shew that the area of the triangle CDE is $\frac{a^2b^2}{2c^2}(\cot\alpha\sim\cot\beta)$.

268. If $ABCD$ be a quadrilateral, whose opposite angles are supplementary, shew that $\tan\frac{A}{2}=\sqrt{\frac{(s-a)\,(s-d)}{(s-b)\,(s-c)}}$; and shew how this may be reduced to the corresponding expression for a triangle, by supposing one of the sides (b, c) opposite to the angle A to vanish.

269. Prove that $\log_e\left(\frac{\cot\frac{1}{2}\vartheta}{\sin\vartheta}\right)=\frac{\cos\vartheta}{1}+\frac{\cos^2\vartheta}{2}+\frac{\cos^3\vartheta}{3}+$ &c.

270. A triangular field is to be divided among three persons in proportion to their ages, which are m, n, p, years respectively, by means of two fences, parallel to one of its sides. Shew how this may be done.

271. Prove the formulæ

$$\frac{\operatorname{cosec}2A}{1+\operatorname{cosec}2A}=\frac{1+\tan^2A}{(1+\tan A)^2}\text{ and }\operatorname{cosec}A=\cot\tfrac{1}{2}A-\cot A.$$

272. Sum *ad infinitum* $(1-3^{-\frac{1}{2}})-\frac{1}{3}(1-3^{-\frac{3}{2}})+\frac{1}{5}(1-3^{-\frac{5}{2}})-$&c.

273. Prove, *a priori*, that $\sin mA$, when expressed in terms of $\sin A$, will have one or two values, according as m is odd or even, and that $\cos mA$, in terms of $\cos A$, will have only one value, m being in each case a positive integer.

274. Find θ when $2+\cot^2\theta=3\sec^4\vartheta-\tan^2\theta$.

275. When $x=\frac{1}{2}\pi$, shew that $\left(x-\frac{2x^2}{\pi}\right)\tan x=1$.

276. A ladder, whose length is 30 feet, stands against a wall, at an angle of 60° with the horizon. At what distance from its top must another of the same length be fastened, at an angle of 75° with the horizon, so as just to reach a window 48 feet from the ground?

277. Solve the equation $8^x.125^{2-x}=2^{4x+3}.5^x$, given log 2=.30103.

278. On the sides of an equilateral triangle three squares are described. Compare the area of the triangle formed by joining the centres of these squares with the area of the equilateral triangle.

279. From the values of BD and $\sin A$ in (116), obtain the expression for the radius of the circle, which circumscribes a given quadrilateral, in terms of its sides.

280. If $\cos 2\theta=4\cos\alpha\cos\theta$, show that $\cos\theta=\cos 45°\tan(\phi\pm 45°)$, where $\phi=\frac{1}{2}\tan^{-1}(\cos\alpha\sec 45°)$.

281. Let r_1, r_2, r_3, be the radii of the circles, which touch one side of a triangle ABC and the other two sides produced: then $r_1\cot\frac{1}{2}A=r_2\cot\frac{1}{2}B=r_3\cot\frac{1}{2}C=\sqrt{(r_1r_2+r_1r_3+r_2r_3)}$.

282. Shew that the ratio of the areas of regular decagons, described in and about the same circle, will be as 7 : 8 nearly, more nearly as 29 : 32, and still more nearly as 123 : 136.

283. Determine $\cos\theta$ and $\cos\phi$ from the equations

$$p\sin^4\theta-q\sin^4\phi=p,\quad p\cos^4\theta-q\cos^4\phi=q.$$

284. From the top of a tower a person observes the depressions (α, β) of two distant points in the horizontal plane at the foot of the tower, (whose distance from each other he knows to be a miles,) and also the angle (θ) subtended at his eye by the line joining the two points. Find the height of the tower, and adapt the expression to logarithmic computation.

285. If the quadrilateral in (116) can also have a circle inscribed within it, shew that the sums of its opposite sides are equal, and thence that its area $=\sqrt{(abcd)}$: determine also the radius of the circle.

286. Shew that $\sin\frac{1}{2}A$, in terms of $\cos A$, will have two values, but in terms of $\sin A$, four values.

287. Calculate the common log of 255 to four places of decimals, having given $\log_{10}2=.30103$, and the *modulus* .4342944819.

288. Given $\sin(\alpha-\theta)=\cos(\alpha+\theta)$, find θ.

289. If $\tan\theta = \frac{\sin\phi}{\sin\psi}\sin(\phi+\psi)$, shew that

$$\tan\phi = \frac{\sin\theta}{\sin(\theta+\phi+\psi)}\sin(\phi+\psi).$$

290. In a given circle is inscribed an equilateral triangle, and in that triangle another circle, and so on *ad inf.*: shew that the sum of the circumferences of all the inner circles is equal to that of the given circle, and the sum of their areas to a third of its area.

291. Find θ from the equation $\sin\theta - \cos\theta = 4\cos^2\theta\sin\theta$.

292. The angles A, B, C, of a triangle are as the numbers 2, 3, 4: shew that $\cos\frac{1}{2}A = \frac{a+c}{2b}$.

293. Three circles, whose radii are a, b, c, touch each other externally: prove that the tangents at the point of contact meet in a point whose distance from any one of them is $\left(\frac{abc}{a+b+c}\right)^{\frac{1}{2}}$.

294. If a circle be inscribed in a square, and between it and the four angular points four other circles be described, and so on continually, shew that the sum of the perimeters of all these circles: that of given circle :: $2\sqrt{2}-2$: 1, and the corresponding areas :: $3\sqrt{2}-4$: 2.

295. Shew that the sum of the squares of the distances of the centre of the inscribed circle from the angular points of a triangle $= ac+bc+ab - \frac{6abc}{a+b+c}$.

296. Adapt the expression $\frac{1}{a+b\cos x}$ to logarithmic computation, both when a is greater and less than b.

297. Prove that $\frac{2}{3}(8\sin\frac{1}{4}\theta - \sin\frac{1}{2}\theta) = \theta$ nearly, if θ be small.

298. AB, AC, are the chords of arcs of 60° and 90° in a circle, whose centre is O: shew that, if AB and OC be produced to meet in D, the arc, whose chord is equal to DC, has its cosine and chord equal.

299. Solve the equation $\tan^{-1}\frac{4(x-x^3)}{1+x^2} = 2\pi - 2\,\mathrm{chd}^{-1}\frac{2}{\sqrt{(1+x^2)}}$.

300. A person, walking from C to D on the horizontal road, can see plainly an obelisk on the summit of a hill at A, at every point but E, where he can *just* see the obelisk over the hill B.

He then measures a base EC (a), and at C observes the elevations (α, β) of A and B, as well as the angles ACB, ACE, (γ, δ). At E he observes the angle AEC (ϵ). Find the heights of the two hills.

301. From the four angles of an inscribed quadrilateral, lines are drawn to the diagonals, making equal angles with them: shew that the product of two of these lines, taken alternately, is equal to the product of the other two.

302. If $\theta+\phi+\psi=90°$, shew that

$(\tan\theta+\tan\phi+\tan\psi)\ (\cot\theta+\cot\phi+\cot\psi)=1+\operatorname{cosec}\theta\ \operatorname{cosec}\phi\ \operatorname{cosec}\psi$.

303. Find θ from the equation $\sin^2 2\theta-\sin^2\theta=\frac{1}{4}$; and express its value by a single formula.

304. Find the circular measure of $5°\ 43'\ 46''.46$; and by means of the formula $\sin\alpha=\alpha-\frac{1}{6}\alpha^3+$&c, obtain the numerical value of the sine of this angle, to seven places of decimals.

305. If $\sin\frac{1}{3}A$ be found from a given value of $\cot A$, shew how many values of $\sin\frac{1}{3}A$ may be expected.

306. Find x in the equation $\sin^{-1}(2x)-\sin^{-1}(x\sqrt{3})=\sin^{-1}x$.

307. If $A+B+C=\pi$, shew that

$$\frac{\tan A}{\tan B}+\frac{\tan B}{\tan C}+\frac{\tan C}{\tan A}+\frac{\tan A}{\tan C}+\frac{\tan C}{\tan B}+\frac{\tan B}{\tan A}$$
$$=\sec A\ \sec A\ \sec C-2.$$

308. P is any point in the circumference of a circle, whose diameter is AB. Take $\angle PAB=\theta$, and find, geometrically, the expression for $2\sin\theta$ in terms of $\sin 2\theta$.

309. If $\dfrac{2}{1+x}=\dfrac{\sin\beta\sin\theta}{\cos(\beta-\alpha)}=\dfrac{\tan(\theta-\alpha)}{\cot\beta}$, shew that

$$x=\{\cot\tfrac{1}{2}\alpha-2\cot\beta\}^{\frac{1}{2}}\ \{\tan\tfrac{1}{2}\alpha+2\cot\beta\}^{\frac{1}{2}}.$$

310. Write down the first twenty positive values of θ, which satisfy the equation $\dfrac{\sin 2\theta}{\theta}\dfrac{\sin 12\theta}{\sin 3\theta}=0$; and express generally the values of θ which satisfy it.

311. If two lines, at right angles to one another on an inclined stratum, make angles α, β, with the horizontal plane, shew that the dip (θ) of the stratum is given by the equation

$$\cos\theta=\{\cos(\alpha+\beta)\cos(\alpha-\beta)\}^{\frac{1}{2}}.$$

312. A circle revolves in its own plane round a point O, and from a point P in that plane the greatest and least angles under which the circle is seen are 2α, 2β. Shew that the distance of O from the centre of the circle$=PO\sin\phi$, where

$$\tan^2(45°+\tfrac{1}{2}\phi)=\sin\alpha\div\sin\beta.$$

313. At what distance may the Peak of Teneriffe, which is $2\frac{1}{2}$ miles above the level of the sea, be first seen from the topmast of a ship, 89 feet above the surface?

314. Given the areas A, B, of the regular inscribed and circumscribed polygons of a circle, each of the same number of sides, find the areas of the corresponding regular polygons of twice the same number of sides.

315. The sides of a triangle are in A. P., and its area is to that of an equilateral triangle of the same perimeter as 3 : 5. Shew that its largest angle is 120°.

316. If the sides of a triangle are a, b, c, and another triangle be formed, whose sides are proportional to $a \cos A$, $b \cos B$, $c \cos C$, shew that the angles of this latter triangle will be the supplements of $2A$, $2B$, $2C$, respectively.

317 Eliminate θ and ϕ from the equations
$a \sin^2\theta + b\cos^2\theta = m$, $a\cos^2\phi + b\sin^2\phi = n$, $a\tan\theta = b\tan\phi$.

318. A staff, one foot long, stands at the top of a tower 200 *feet* high. Shew that the angle which it subtends at a point in the horizontal plane, 100 *feet* from the base of the tower, is 6′ 51″, nearly.

319. Shew that in any triangle $\tan(B + \frac{1}{2}A) = \frac{c+b}{c-b}\tan\frac{1}{2}A$; and if $3c = 7b$, and $A = 6°\ 37'\ 24''$, find the other angles, given
$\log 2 = .30103$, L $\tan 3°\ 18'\ 42'' = 8.7624080$,
L $\tan 8°\ 14' = 9.1604569$, L $\tan 8°\ 13'\ 50'' = 9.1603083$.

320. Given $\tan\theta + \tan\phi + \tan\psi = 1 + \frac{2}{3}\sqrt{3}$, $\tan\theta\tan\phi\tan\psi = 1$, and $\tan\theta\tan\phi + \tan\theta\tan\psi + \tan\phi\tan\psi = 1 + \frac{2}{3}\sqrt{3}$; find θ, ϕ, ψ.

321. Given $\tan\frac{1}{2}\alpha = \tan^3\frac{1}{2}\beta$, and $\tan\beta = 2\tan\phi$, shew that ϕ is an arithmetic mean between α and β.

322. Two objects are seen in the same direction to the SE, and, when the observer has moved 8 miles due S, the one bears NE, and the other due E: how far are they apart?

323. If the tangents of the angles, A, B, C, of a triangle increase in a G. P., whose ratio is n, shew that $\sin 2A = n \sin 2C$.

324. Find x from the equation $\sec^{-1}a + \sec^{-1}\frac{x}{a} = \sec^{-1}b + \sec^{-1}\frac{x}{b}$.

325. If in a regular polygon lines be drawn from the extremities of any side (a) to those of any other side, so as to cross each other, shew that their intersections will all lie in a circle whose radius is $\frac{a}{2}\operatorname{cosec}\frac{2\pi}{n}$.

326. In the ambiguous case of oblique triangles, if c, c', be the two values found for the third side of the triangle, shew that
$$c^2+2cc'\cos 2B+c'^2=4b^2\cos^2 B.$$

327. If r, r_1, r_2, r_3, be the radii of the inscribed and escribed circles of a triangle, and R of its circumscribing circle, shew that the area of the triangle $=\sqrt{(rr_1r_2r_3)}$, and that
$$r_1+r_2+r_3-r=4R.$$

328. Shew that the triangle, square, and hexagon, are the only regular figures by which space can be completely filled up about a point.

329. Given $\sin(\theta+\phi)=\tan(\theta-\phi)=\frac{1}{2}(\tan\theta-\tan\phi)$, find θ and ϕ.

330. Find x from the equation
$$\operatorname{vers}^{-1}(1+x)-\operatorname{vers}^{-1}(1-x)=\tan^{-1}2\sqrt{(1-x^2)}$$

331. Given that the line, bisecting the angle A (also given) of a triangle, divides the base in the ratio of $m:n$, find $\tan B$, and adapt the result to logarithmic computation.

332. If the angle C of a triangle be very obtuse, shew that its defect from $180°=206265\sqrt{\frac{2(a+b)(a+b-c)}{ab}}$ seconds, nearly.

333. Two objects, P, Q, are observed from a ship to be at the same instant in a line bearing N. 15° E. The ship sails NW for 5 miles, and then P bears due E, and Q bears NE. Find the distance between P and Q.

334. The perimeter of a triangle is four times its base (c): shew that if r be the distance of the centre of its inscribed circle from one end of the base, and ϕ the angle it makes with the base, then $r(1+\sin^2\phi)=c\cos\phi$.

335. If $r=\tan\{\sin^{-1}(\frac{\sin z}{1+n})\}$, where n is very small, prove that $r=\tan\{z(1-n\sec^2 z)\}$, nearly.

336. The distance between the centres of two wheels is a, and the sum of their radii is c: shew that the length of the string, which crosses between the wheels and just wraps around them, is
$$2\{\sqrt{(a^2-c^2)}+c\cos^{-1}(-\tfrac{c}{a})\}.$$

337. Given $a = \sec\theta + \operatorname{cosec}\theta \tan^3\theta\,(\operatorname{cosec}^2\theta + 1)$
$b = \tan\theta - \tan^3\theta\,(\operatorname{cosec}^2\theta + 1)$;
shew that $a^{\frac{2}{3}} - b^{\frac{2}{3}} = 2^{\frac{2}{3}}$.

338. A person standing at known distances (a, b) from two towers, in the same vertical plane with himself, observes their apparent altitudes to be the same. He then walks a given distance (c) towards the towers, till the elevation of the one is double that of the other. Find their heights.

339. The alternate angles of a regular pentagon being joined, shew that the area of the interior pentagon thus formed is to that of the original figure as $3+\sqrt{5} : 3-\sqrt{5}$.

340. In the figure of *Euc.* I. 1, find the area of the part common to both the circles, the radius of each being a.

341. A piece of paper in the form of a circular sector, whose angle is 2 and radius 2 inches, is formed into a conical cap. Find the area of the conical surface, and also of the base of the cone.

342. Shew that x cannot be real in the equation $\tan^3 x = \tan(x-a)$, if $\sin a$ be greater than $\frac{1}{3}$.

343. Given $\tan^2 x + \sec 2x = 7 - \frac{10}{3}\sqrt{3}$, find x.

344. From the extremity A of the diameter ACB of a semicircle, whose centre is C, a line AP is drawn to the circumference, bisecting the semicircle. If θ be the circular measure of the complement of PCB, prove that $\cos\theta = \theta$.

345. Two circles have a common radius r, and a circle is described touching this radius and the two circles: prove that the radius of the circle which touches the three $= \frac{3}{56}r$.

346. Determine x and y from the equations
$\tan^{-1}x - \tan^{-1}y = \cot^{-1}2y - \cot^{-1}2x = \frac{1}{4}\pi$.

347. Shew that $\frac{1}{3}\{2(\sin 2\theta - \sin\theta) + (\tan 2\theta - \tan\theta)\} = \theta$, very nearly, if θ be small.

348. A ship, sailing SSW, was observed by another, sailing away from the former, due S, to subtend an angle of 20′: its length being known to be 150 feet, determine its distance.

349. The sides AB, BC, &c. of an inscribed quadrilateral are in G. P., of which the common ratio is m: shew that
$\tan ABC : \tan BCD :: m^2 - 1 : m^2 + 1$.

350. If $\tan a \tan x = \tan^2(a+x) - \tan^2(a-x)$, find $\cos x$.

351. If two semicircles be described upon the bounding radii of a quadrant, the circle which touches the three circumferences will have its radius : radius of quadrant:: $\sqrt{2}-1 : 3\sqrt{2}-1$.

352. A circle is inscribed in a triangle, and a second triangle is formed, whose sides are equal to the distances of the points of contact from the angles of the triangle. If r be the radius of the circle inscribed in the first triangle, and r', r'', the radii of the inscribed and circumscribed circles of the second triangle, then will $r^2=2r'r''$.

353. The area of a regular pentagon : area of isosceles triangle used in its construction (*Euc.* IV. 11):: $\sqrt{5} : 1$.

354. Determine x, when

$$2\operatorname{cosec}^{-1}\sqrt{(x^2-2x+2)}=\tfrac{1}{2}\pi+\operatorname{chd}^{-1}\frac{2}{\sqrt{(x^2+2x+2)}}.$$

355. A hexagon is formed by drawing lines from each angle of a triangle, perpendicular to the sides containing that angle. Shew that a circle may be described about it, and that the area of the hexagon is double of that of the triangle.

356. Shew that in the ambiguous case, when a and b are very nearly equal, the number of seconds in the angle C will be (nearly) $\dfrac{(a-b)\tan B}{b\sin 1''}$ or $(180-2B)\times 60\times 60-\dfrac{(a-b)\tan B}{b\sin 1''}$.

357. Solve the equations

$$\theta-\varphi=\sin^{-1}\frac{1}{\sqrt{10}},\quad \frac{\sin^2\theta\cos^2\phi+\cos^2\theta\sin^2\phi}{\sin^2\theta\cos^2\varphi-\cos^2\theta\sin^2\varphi}=\frac{5}{3}.$$

358. In any triangle, given the sides b, c, shew that if A receive a small increment (a), then B and C will receive small decrements, respectively,

$$a\sin B\cos C\operatorname{cosec}A, \text{ and } a\sin C\cos B\operatorname{cosec}A.$$

359. Prove geometrically the expressions for $\sin\theta$ in terms of $\cos 2\theta$ and $\sin 2\theta$; and explain the two values in the former case, and the four in the latter.

360. The angles of a triangle are as 3, 4, 5 : find its area, having given r, the radius of its inscribed circle.

361. Given

$$\sin\theta\sqrt{(1+\tan^2\alpha\tan^2\beta)}+\cos\theta\sqrt{(1-\tan^2\alpha\tan^2\beta)}=\tan\alpha+\tan\beta,$$

find θ in a form adapted to logarithmic computation.

362. If the middle points of the sides of a triangle be joined with the opposite angles, and r_1, r_2, &c. R_1, R_2, &c. be the radii of

the circles described *in* and *about* the six triangles thus formed, shew that

$$R_1R_3R_5 = R_2R_4R_6 \text{ and } \frac{1}{r_1}+\frac{1}{r_3}+\frac{1}{r_5}=\frac{1}{r_2}+\frac{1}{r_4}+\frac{1}{r_6}.$$

363. If from the ends of the base of a semicircle two chords be drawn, crossing each other and making angles α, β, with the base, and if the extremities of these chords be joined, the areas of the two triangles thus formed will be as $\cos^2(\alpha+\beta) : 1$.

364. Shew that $\sin(45^\circ \pm x) = \sqrt{\{\frac{1}{2}(1 \pm \sin 2x)\}}$; and thence prove that $2\cos\left(60^\circ \pm \frac{30^\circ}{2^n}\right) = \sqrt{[2-\sqrt{\{2-\sqrt{(2-\&c)}\}}]}$, where the (2) occurs n times, and the upper or lower sign is to be used, according as n is even or odd.

365. If θ be a small angle, containing n'', shew that
$$\log_{10} n = \mathrm{L}\sin n'' + \tfrac{1}{3}(10 - \mathrm{L}\cos\theta) - \mathrm{L}\sin 1'', \text{ nearly.}$$

366. Eliminate θ from the equations
$$a\tan\theta + b\sec\theta = c, \quad a\cot\theta + b\operatorname{cosec}\theta = d.$$

367. If, in any triangle ABC, a line be drawn from A to the point D, where the inscribed circle touches the opposite side, and if in the two triangles, thus formed, circles be inscribed, shew that they will touch AD in the same point.

368. If $\theta+\phi+\psi = 90^\circ$, shew that $\sin\theta + \sin\phi + \sin\psi - 1 = \sqrt{2}\,(\cos\frac{1}{2}\theta - \sin\frac{1}{2}\theta)\,(\cos\frac{1}{2}\phi - \sin\frac{1}{2}\phi)\,(\cos\frac{1}{2}\psi - \sin\frac{1}{2}\psi)$.

369. The distance between two points is c, and their distances from a given line a, b. Of all triangles that can be formed, having the same base c, and with their vertices in the given line, the area of that which has the least perimeter is
$$\frac{ab}{a+b}\sqrt{\{(c+a-b)\,(c+b-a)\}}.$$

370. From a vessel sailing SW at the rate of 8 miles an hour a landmark is seen to bear SE by E, and $1\frac{1}{2}$ hour afterwards it bears E by N. Determine the distance of the vessel from the landmark at each observation, given
$$\sin 33^\circ\,45' = .555, \quad \cos 11^\circ\,15' = .980.$$

371. If from any point P, within or without a triangle, perpendiculars Pa, Pb, Pc, be dropped upon the sides BC, AC, AB, and circles be described about the triangles Pab, Pac, Pbc, the area of the triangle, formed by joining the centres of these circles, will be $\frac{1}{4}$ of the area of the triangle ABC.

372. Find the values of x which satisfy the equation
$$\sin\{2\cos^{-1}\cot(2\tan^{-1}x)\}=0.$$

373. If $a=b+c$, shew that
$$\left\{\frac{1+\cos ma}{ab}-\frac{1-\cos mc}{bc}\right\}^2+\left\{\frac{\sin ma}{ab}-\frac{\sin mc}{bc}\right\}^2$$
$$=\frac{1}{abc}\left\{\frac{a^2+b^2+c^2}{abc}+2\frac{\cos ma}{a}-2\frac{\cos mb}{b}-2\frac{\cos mc}{c}\right\}:$$
and find the value of either side, when $a=c$.

374. If θ be the angle between the diagonals of a parallelogram, whose sides (a, b) are inclined at an angle (α), find $\tan\theta$.

375. Six circles are described between the escribed circles of a triangle and the angular points, each touching a side and a side produced. Shew that the products of their radii, taken alternately, are each equal to $Ss\tan^2\frac{1}{4}A\tan^2\frac{1}{4}B\tan^2\frac{1}{4}C$, where S is the area of the triangle, and $2s=a+b+c$.

376. If $\cos^2\theta=\frac{\cos\alpha}{\cos\beta}$, $\cos^2\theta'=\frac{\cos\alpha'}{\cos\beta}$, and $\frac{\tan\theta}{\tan\theta'}=\frac{\tan\alpha}{\tan\alpha'}$, shew that $\tan\frac{1}{2}\alpha\tan\frac{1}{2}\alpha'=\tan\frac{1}{2}\beta$.

377. Shew that
$$\tfrac{1}{4}a^3\operatorname{cosec}^2(\tfrac{1}{2}\tan^{-1}\tfrac{a}{b})+\tfrac{1}{2}b^3\sec^2(\tfrac{1}{2}\tan^{-1}\tfrac{b}{a})=(a+b)(a^2+b^2).$$

378. Prove that $\tan\frac{1}{4}A$, when expressed in terms of $\sin A$, will have four different values.

379. If $\cot\theta-\cot\alpha=2\sec 2\phi$, and $\cot\theta'-3\sin\alpha'=\sin^2\phi$, find what values must be given to α and α', so that $\tan\theta-\tan\theta'$ may have a fixed numerical value for all values of ϕ.

380. If a, a', be homologous sides of similar triangles, described *in* and *about* a circle, shew that $a=4a'\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C$.

381. The elevations (α, β, γ) of a balloon are taken at the same moment, at three points A, B, C, in the same horizontal line. Given $AB=a$, $BC=b$, determine its altitude.

382. Given $\operatorname{vers}^{-1}\frac{x}{a}-\operatorname{vers}^{-1}\frac{bx}{a}=\operatorname{vers}^{-1}(1-b)$: find x.

383. A person, wishing to determine the length of a wall upon the opposite side of a stream, places himself due South of one end, and then due West of the other, at such distances that the angle which the wall subtends at each position is (α). Shew that, if a be the distance between the two stations, the length of the wall will be $a\tan\alpha$.

384. If $\cos(a-x) = \cos a \cos b$, where x is very small, prove that $x = -2 \cot a \sin^2 \frac{1}{2}b (1 - \cot^2 a \sin^2 \frac{1}{2}b)$, very nearly.
How may we find from this the number of seconds in x?

385. If $m \cos\phi = \sqrt{\{\frac{1}{3}(1-m^2)\}}$ and $\tan\alpha = \tan^3 \frac{1}{2}\phi$, shew that
$$m = \tfrac{1}{2}(\cos^{\frac{2}{3}}\alpha + \sin^{\frac{2}{3}}\alpha)^{\frac{3}{2}}.$$

386. Two ships are sailing uniformly, with velocities u, v, along lines inclined at an angle θ. Given a, b, their distances at the same moment from the point of intersection of their courses, find their least distance from each other.

387. On the sides of a triangle ABC, as diameters, three circles are described intersecting in P, Q, R: shew that the perimeter of ABC: that of PQR:: 1 : 4 $\sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C$.

388. Find the value of x, when $\sin 2x = \sin^2 3x$.

389. A and B are two observed angles of a plane triangle, which have small unknown errors. What is the form of the triangle which will give $\sin(A+B)$ with the least error?

390. Shew that θ may be found from the equation
$$\sin^3\theta = \sin(\alpha-\theta)\sin(\beta-\theta)\sin(\gamma-\theta),$$
where $\alpha+\beta+\gamma = \pi$, by either of the formulæ
$$\cot\theta = \cot\alpha + \cot\beta + \cot\gamma, \text{ or } \operatorname{cosec}^2\theta = \operatorname{cosec}^2\alpha + \operatorname{cosec}^2\beta + \operatorname{cosec}^2\gamma.$$

391. A circle is described about a triangle ABC, and a new triangle $A'B'C'$ is formed by joining the points of bisection of the arcs, subtended by the sides of ABC. Shew that the sides of this triangle are $\frac{1}{2}a \operatorname{cosec} \frac{1}{2}A$, $\frac{1}{2}b \operatorname{cosec} \frac{1}{2}B$, $\frac{1}{2}c \operatorname{cosec} \frac{1}{2}C$, and that its area : area ABC:: 1 : 8 $\sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C$.

392. The shadows of two vertical walls, which are at right angles to each other, and are a and a' feet in height, are observed, when the Sun is due South, to be b and b' feet in breadth. Shew that, if α be the Sun's altitude, and β the inclination of the first wall to the NS line (or *meridian*,) then
$$\cot\alpha = \sqrt{(\frac{b^2}{a^2} + \frac{b'^2}{a'^2})}, \quad \cot\beta = \frac{ab'}{a'b}.$$

393. The circles described through any three of the centres of the four circles, that can be drawn touching the sides or sides produced of a triangle, have all the same radius, which is twice that of the circumscribing circle of the triangle.

394. If P be the centre of a circle, radius R, which passes through the three angles of a triangle, and Q the centre of a circle,

radius r, which touches the three sides, then $PQ = \sqrt{(R^2 \mp 2Rr)}$, $\mp$ according as the latter circle lies *within* or *without* the triangle.

395. If Q_1, Q_2, Q_3, Q_4, be the centres of the inscribed and escribed circles of a triangle, and P that of the circumscribing circle, shew that $PQ_1^2 + PQ_2^2 + PQ_3^2 + PQ_4^2 = 12R^2$.

396. Prove that $\sin\theta = \theta \cos\frac{\theta}{2} \cos\frac{\theta}{2^2} \cos\frac{\theta}{2^3} \ldots$ *ad infinitum*.

397. Find the value of $\frac{1}{\sqrt{x}} \operatorname{vers}^{-1}\frac{x}{a}$, when $x = 0$.

398. At noon, a column in the ESE cast upon the ground a shadow, the extremity of which was in the direction NE. The elevation of the column, as seen from the spectator, being (α), and his distance from the extremity of the shadow a feet, determine the height of the column.

399. Find the value of $(1 + \sec\theta)\left(1 + \sec\frac{\theta}{2}\right) \ldots \left(1 + \sec\frac{\theta}{2^{n-1}}\right)$.

400. The sides of a triangle are in A.P., and the distance of the centres of the inscribed and circumscribed circles is a mean proportional between the greatest and least of them. Shew that the sides are as the numbers $\sqrt{5}-1$, $\sqrt{5}$, $\sqrt{5}+1$.

401. If on the sides of a triangle are described three triangles, whose vertical angles are equal to those angles of the triangle, which are respectively opposite to them, shew that the triangle, formed by joining the centres of the circles circumscribed about them, is in every respect equal to the original one.

402. Find the value of θ, when
$$(1 + \cos\alpha\cos\theta)\tan(\theta + \alpha) + (\cos\alpha + \cos\theta)\sin\alpha = 0.$$

403. If $\alpha < 45°$, prove that
$$2\tan^{-1}\{\tan\tfrac{1}{2}45° - \alpha)\tan\tfrac{1}{2}\beta\} = \cos^{-1}\frac{\tan\alpha + \cos\beta}{1 + \tan\alpha\cos\beta}.$$

404. A flag-staff, 8 feet high, on the top of a tower, subtends an angle of $57'\ 17''.75$ at 100 yards from the foot of the tower: determine its height.

405. If x, y, z, be the lengths of lines, bisecting the angles of a triangle, and terminated by the opposite sides, a, b, c, shew that
$$(b+c)^2\frac{x^2}{bc} + (a+c)^2\frac{y^2}{ac} + (a+b)^2\frac{z^2}{ab} = (a+b+c)^2.$$

406. Find the area between three circles which touch one another, and whose diameters are $\sqrt{3}-1$, $\sqrt{3}+1$, $3-\sqrt{3}$, $(\pi=\frac{22}{7})$.

407. The base of a pyramid being square, and its sides equilateral triangles, if θ, ϕ, be the inclinations of its sides to the base and to each other respectively, then $\cos^2\theta=\cos(\pi-\phi)$.

408. Shew that, if $\alpha+\beta+\gamma=90°$, then

$$\frac{\cos\alpha+\sin\gamma-\sin\beta}{\cos\beta+\sin\gamma-\sin\alpha}=\frac{1+\tan\frac{1}{2}\alpha}{1+\tan\frac{1}{2}\beta}.$$

409. If the circumference of a circle be divided into arcs of 60° in the points P_1, P_2, &c, and if P_1P_3 be joined, cutting the radius OP_2 in Q_1, and then Q_1P_4 be joined, cutting OP_3 in Q_2, and so on, then OQ_1, OQ_2, OQ_3, &c. are respectively the half, third, fourth, &c. parts of the radius.

410. In a segment of a circle, whose radius is r, and length of arc l, a circle is described, touching the middle point of the arc. Shew that the radius of the circle, which touches the chord and arc of the segment, and also the inscribed circle, is $\frac{r}{4}\sin^2\frac{l}{2r}$.

411. Prove that $\tan^3\frac{1}{4}\{\sin^{-1}(3\sin\alpha)+\alpha\}=\tan\frac{1}{4}\{\sin^{-1}(3\sin\alpha)-3\alpha\}$.

412. Solve the equation $\cos 7\theta+7\cos\theta=0$.

413. If A, B, C, be the angles of a triangle, shew that

$$\cos nA+\cos nB+\cos nC=1\pm4\sin\tfrac{1}{2}nA\sin\tfrac{1}{2}nB\sin\tfrac{1}{2}nC,$$
$$\text{or}=-1\pm4\cos\tfrac{1}{2}nA\cos\tfrac{1}{2}nB\cos\tfrac{1}{2}nC,$$

according as n is odd or even.

414. The centres of the escribed circles of a triangle are joined, so as to form another triangle; this is similarly treated, and so on. Shew that the radii of the circumscribing circles will form a G. P., whose ratio is 2.

415. From the equation $h\cos(\theta+\alpha)+\cos\alpha=0$ derive the equation

$$(1+h\cos\theta)\tan(\theta+\alpha)+(h+\cos\theta)h\tan\alpha=0.$$

416. Shew that $2\cos\frac{45°}{2^{n-1}}=\sqrt{\{2+\sqrt{(2+\&c)}\}}$ to n terms.

417. Squares are described upon the sides of a triangle ABC, and their centres are joined, forming the triangle LMN: prove that area $LMN=(1+\frac{1}{2}p)$ area ABC,

where $p=\cot A+\cot B+\cot C$

418. On the side of a hill there are two places, B, C, inaccessible to each other, but known to be at the same distance (a) from the same station A. At the lower place C, the horizontal angle (α) between A and B is measured, as well as their altitudes (λ, μ). Shew that the distance BC

$$=2a\{\cos(\lambda-\mu)\cos^2\tfrac{1}{2}\alpha-\cos(\lambda+\mu)\sin^2\tfrac{1}{2}\alpha\}.$$

419. If $\tan\theta=\dfrac{b}{a}$, shew, by Demoivre's Theorem, that

$$(a+b\sqrt{-1})^{\frac{m}{n}}+(a-b\sqrt{-1})^{\frac{m}{n}}=2(a^2+b^2)^{\frac{m}{2n}}\cos\frac{m}{n}\theta.$$

420. Shew that if $2\cos\alpha_n=x_n+\dfrac{1}{x_n}$, and $\alpha_1+\alpha_2+\&c+\alpha_n=2\pi$, then $x_1x_2x_3\ldots x_n=1$.

421. If $a^2+b^2=2$ and $ab=\sqrt{-1}\tan 2\phi$, then $a=\cos\phi\sqrt{(2\sec 2\phi)}$.

422. If $\theta=\cos\theta$, shew that θ contains $42°\ 20'$, nearly.

423. Find the sum of n terms of the series

$$\tan^{-1}a+\tan^{-1}\frac{a}{1+1.2a^2}+\tan^{-1}\frac{a}{1+2.3a^2}+\&c.$$

424. If $e^x\cos x$ be expanded in a series of ascending powers of x, find the coefficient of x^n.

425. Sum to n terms the series $\sin^2\theta+\sin^2 2\theta+\sin^2 3\theta+\&c.$

426. Prove that $\sin 5°\sin 15°\sin 25°\ldots\sin 85°=\dfrac{1}{256\sqrt{2}}$.

427. Find by Trigonometry the roots of $x^3-3x+1=0$.

428. Prove that the sum *ad infinitum* of the series, whose n^{th} term is $(-1)^{n-1}\dfrac{n}{2n-1}\cdot\dfrac{1}{\lfloor 2(n-1)}$, is $\dfrac{1}{\sqrt{2}}\sin\left(\dfrac{\pi}{4}+1\right)$.

429. Given $\theta=\cot\theta$, shew that $\theta=49°\ 17'$ nearly.

430. Shew that $\sin 10°\sin 30°\sin 50°\sin 70°=\frac{1}{16}$.

431. If l be the length of a curve BC upon a line of railroad, and c the direct distance between its extreme points, shew that the radius AB of the circle of which the curve is a portion is $\dfrac{1}{2}\sqrt{\dfrac{l^3}{6(l-c)}}$, nearly.

432. Find by Trigonometry the roots of $2x^3-3x-1=0$.

433. If $\tan\alpha=m\tan\beta$, shew that

$$\alpha=\beta+\frac{m-1}{m+1}\sin 2\beta+\frac{1}{2}\left(\frac{m-1}{m+1}\right)^2\sin 4\beta+\&c.$$

434. Shew that $x\sin\theta+\frac{x^2}{2}\sin 2\theta+\frac{x^3}{3}\sin 3\theta+\&c=\tan^{-1}\left(\frac{x\sin\theta}{1-x\cos\theta}\right)$.

434. Solve the equation $8x^3-36x^2+42x-13=0$ by Trigonometry.

436. If m be any integer, prove that

$$\tan\frac{\pi}{4m}\tan\frac{5\pi}{4m}\tan\frac{9\pi}{4m}\ldots\tan\frac{(4m-3)\pi}{4m}=\pm 1.$$

537. Shew that $\cos 2\theta\ \mathrm{cosec}^2 2\theta=\frac{1}{2}\mathrm{cosec}^2 2\theta-\mathrm{cosec}^2 2\theta$:
and thence sum to n terms the series
$\cos 2\vartheta\ \mathrm{cosec}^2 2\vartheta+2\cos 4\theta\ \mathrm{cosec}^2 4\vartheta+4\cos 8\theta\ \mathrm{cosec}^2 8\vartheta+\&c.$

438. If $\tan(\frac{1}{4}a-\phi)=\tan^3\frac{1}{4}a$, shew that

$$\phi=\frac{1}{1.3}\sin a-\frac{1}{2.3^2}\sin 2a+\frac{1}{3.3^3}\sin 3a-\&c.$$

439. Sum to n terms the series $\sin^3\vartheta+\sin^3 2\vartheta+\sin^3 3\theta+\&c.$

440. In any triangle shew that $\log_e b-\log_e a$
$=(\cos 2A-\cos 2B)+\frac{1}{2}(\cos 4A-\cos 4B)+\frac{1}{3}(\cos 6A-\cos 6B)+\&c.$

441. Sum to n terms the series
$(\cos\alpha+\sqrt{-1}\sin\alpha)+(\cos\alpha+\sqrt{-1}\sin\alpha)^2+\&c$;
and deduce from it the sums of
$\cos\alpha+\cos 2\alpha+\&c$, and $\sin\alpha+\sin 2\alpha+\&c$.

442. Sum the series $\cos A+\cos 3A+\cos 5A+\&c.$ to n terms.

443. If $\sin\theta=\sin A\sin(B+\theta)$, prove that
$\theta=\sin A\sin B+\frac{1}{2}\sin^2 A\sin 2B+\frac{1}{3}\sin^3 A\sin 3B+\&c.$

444. Sum the series *ad infinitum*

$$\cos\theta+\frac{\cos\theta}{1}\cos 2\theta+\frac{\cos^2\theta}{1.2}\cos 3\vartheta+\frac{\cos^3\theta}{1.2.3}\cos 4\theta+\&c.$$

445. Prove that

$$\tan\frac{\pi}{2^{x+1}}(2^{x-1}+\tan\frac{\pi}{2^{x+1}}+2\tan\frac{\pi}{2^x}+\ldots+2^{x-2}\tan\frac{\pi}{2^3})=1.$$

446. Prove that
$\log_e(1-n\cos\theta)=2(\log_e a-m\cos\theta-\frac{1}{2}m^2\cos 2\theta+\frac{1}{3}m^3\cos 3\theta-\&c)$,
where $m=\frac{\beta}{\alpha}$, α, β, being roots of $2x^2-2x\sqrt{(1+n)}+n=0$.

447. If $x=\cot z$, shew that $\tan^{-1}(x+h)-\tan^{-1}x$
$=h\sin z.\sin z-\frac{1}{2}h^2\sin^2 z.\sin 2z=\frac{1}{3}h^3\sin^3 z.\sin 3z+\&c.$

448. Sum to n terms the series

$$\frac{\sec\theta\sec 3\vartheta}{\mathrm{cosec}\,2\theta}+\frac{\sec 3\theta\sec 5\vartheta}{\mathrm{cosec}\,4\theta}+\frac{\sec 5\theta\sec 7\theta}{\mathrm{cosec}\,6\theta}+\&c.$$

449. If $\tan 2\theta=\cos\alpha\tan 2\phi$, shew that
$\theta=\cos\alpha\tan\phi-\frac{1}{3}\cos 3\alpha\tan^3\phi+\frac{1}{5}\cos 5\alpha\tan^5\phi-\&c.$

450. If from one of the angles of a regular polygon of n sides, lines be drawn to all the other angles, shew that their sum

$= \frac{1}{2} a \operatorname{cosec}^2 \frac{\pi}{2n}$, a being the length of a side.

451. Expand $\log_e (a \cos^2\theta + b \sin^2\theta)$ in a series of the form

$$A + B \cos^2\theta + C \cos^2 2\theta + D \cos^2 3\theta + \&c.$$

452. The circumference of the inner of two concentric circles, radii R, r, is divided into n equal parts. Shew that the sum of the squares of the lines drawn from the points of division to any point in the outer circumference $= n(R^2 + r^2)$.

453. Prove that

$$\sin\theta \sin\theta - \tfrac{1}{2}\sin 2\theta \sin^2\theta + \tfrac{1}{3}\sin 3\theta \sin^3\theta - \&c = \cot^{-1}(1 + \cot\theta + \cot^2\theta).$$

454. If n be an even integer, shew that

$$\operatorname{cosec}^2 \frac{\pi}{2n} + \operatorname{cosec}^2 \frac{3\pi}{2n} + \&c + \operatorname{cosec}^2 \frac{(n-1)\pi}{2n} = \frac{n^2}{2}.$$

455. If C be the middle point of a semicircular arc whose diameter is AB, and the angle ACB be divided into $n+1$ equal angles by lines terminated by the diameter; shew that the sum of the squares of the reciprocals of the dividing lines

$= \frac{1}{2r^2}\{n + \cot \frac{\pi}{2(n+1)}\}$, where r is the radius of the circle.

456. The circumference of a circle, whose centre is C and radius a, is divided into n parts, each of which subtends the same angle at a point O within the circle. If $CO = b$, and r_1, r_2, &c. be the lines from O to the points of division, shew that

$$r_1 + r_2 + \&c = (a^2 - b^2)\left(\frac{1}{r_1} + \frac{1}{r_2} + \&c.\right)$$

457. Circles are inscribed in triangles, whose bases are the sides of a regular polygon of n sides, and whose vertices lie in one of the angular points: shew that the sum of the areas of the circles

$$= 16\pi r^2 \sin^2 \frac{\pi}{2n}\left\{\frac{n}{4}\sin^2 \frac{\pi}{2n} + \frac{n-4}{8}\right\},$$

r being the radius of the circle circumscribing the polygon.

ANSWERS TO THE EXAMPLES.

2. a. 4. $\frac{n+2p}{m+p}$. 6. $\frac{1}{2}ab$. 9. 30 *sq. yds.* 10. .00000042366.

13. $\frac{1}{4}\{\cos(m+n+r)\theta+\cos(m+n-r)\theta+\cos(m+r-n)\theta$
$+\cos(n+r-m)\theta\}$. 17. $\pm\frac{\sqrt{5}+1}{2\sqrt{2}}$. 18. 93.97 *ft.*

19. $\sin A=\frac{1}{2}\{\sqrt{(1-\sin 2A)}-\sqrt{(1+\sin 2A)}\}$,
$\cos A=-\frac{1}{2}\{\sqrt{(1+\sin 2A)}+\sqrt{(1-\sin 2A)}\}$.

26. $-\frac{1}{4}(\sqrt{6}+\sqrt{2})$, $\sqrt{6}+\sqrt{2}$, $-(2+\sqrt{3})$.

31. 30° or $(3n+1)\frac{1}{6}\pi$. 32. 121 *ft.* 35. 7958 *miles.*

38. 35 *sq. yds.* 39. 86.4 *yds*, 115.2 *yds*, 64.8 *yds.*

40. $A=71°\ 15'=79^g\ 16'\ 16''$; $B=3°\ 45'=4^g\ 16'\ 66''$.

41. $\frac{2}{5}\pi$. 42. .8, $\frac{7}{4}$, .894. 45. 121½ *sq. yds.*

47. 47° 44′ 47″. 49. 21 *sq. ft.* 50. 78A 1R 13P.

56. $\frac{1}{2}a\tan a$. 57. 15A 0R 33½P. 61. $\frac{5}{8}\pi$.

63. $\frac{2\sqrt{(ab)}}{a+b}$, $\frac{2b}{a+b}$. 64. 1.8573326, $\bar{1}$.6532126, 3.2552726, $\bar{4}$.3802113.

68. $\frac{a\sin\alpha\sin\beta}{\sin\alpha-\sin\beta}$. 72. .01332556. 77. 45°. 78. 10 and 9.

79. $a^{\frac{1}{3}}b^{\frac{1}{3}}\sqrt{(a^{\frac{2}{3}}+b^{\frac{2}{3}})}$; $(a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{3}{2}}$. 80. .016297; 56′ 1½″.

81. 171° 53′ 14″. 82. 8A 1R 24P. 85. 25 : 4 : 9.

86. $\frac{1}{20}(10m+9n)$ *degrees*; $\frac{1}{18}(10m-9n)$ *grades.* 88. 75 *ft.*

89. $\frac{m-n}{m+n}$. 90. 45° or $(4n+1)\frac{1}{4}\pi$. 92. 31 *sq. in.*

93. 2R 8P 21 *sq. yds.* 94. 90°. 96. 32½ *ft.*

97. 15 *miles.* 99. 96°, 216° : 24°, 36°, 120°. 100. 2640.

101. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=2^{\frac{4}{3}}$. 103. $\pm\cos\frac{1}{2}A$, $\pm\sin\frac{1}{2}A$.

108. 59° 37′ 42″.18; 9.9358987.

110. $(-1)^n\cos\theta$, $(-1)^{n+1}\sin\theta$, $-\cot\theta$.

111. 60° or $\{3n+(-1)^n\}\frac{1}{3}\pi$. 113. $3\frac{3}{14}°$; $173\frac{4}{7}°$.

122. 30° or $(3n+1)\frac{1}{6}\pi$.

124. 90° or $(4n\pm1)\frac{1}{2}\pi$; $\frac{60°}{n-1}$ or $\frac{6n\pm1}{3(n-1)}\pi$.

126. $\bar{1}$.8494850, $\bar{1}$.8409595, $\bar{1}$.8316166.

128. $\frac{2a+b}{\sqrt{3}}$. 130. .6989700, $\bar{2}$.2041200, $\bar{1}$.6989700, .7958800, .2041200, 1.1938200.

131. $\pm(\sqrt{3}+1)$. 134. 6. 135. $a^2-2ac\cos 2\phi+c^2=b^2$.

136. 295 *ft.* 137. $(n+\frac{1}{4})\,\pi-\frac{1}{2}a$. 138. $\sqrt{2}-1$.

139. The limits are $\pm\sqrt{2}$. 140. $\frac{1}{2}ac\sec^2\theta$, where $\tan^2\theta=\frac{b}{a}$.

141. $\frac{1}{2}(\sqrt{5}-1)$. 142. 40°, 80°, 140°, 100°.

144. 15° or $\{6n+(-1)^n\}\,\frac{1}{12}\pi$. 147. $\frac{a\sin\alpha\sin\beta}{\sin(\alpha-\beta)}$.

148. $(4n\pm1)\,\frac{1}{2}\pi$. 150. $m^{\frac{2}{3}}n^{\frac{2}{3}}(m^{\frac{2}{3}}+n^{\frac{2}{3}})=1$.

153. (i) $\frac{2}{3}n\pi$ or $2n\pi$; (ii) $n\pi$ or $(6n\pm1)\,\frac{1}{6}\pi$. 155. 1657 *ft.*

157. $n\pi$ or $n\pi+\alpha$. 159. 3091.230. 161. $339\frac{3}{7}$ *sq. in.*

162. $\pm30°$ or $(6n\pm1)\,\frac{1}{6}\pi$. 164. 44 *ft.*

165. $\pm45°$ or $(4n\pm1)\,\frac{1}{4}\pi$. 168. 55° 46′ 16″. 169. $\sqrt{(4+2\sqrt{2})}$.

171. $k=\sqrt{(a^2+2ab\,\cos\alpha+b^2)}$, $\beta=\tan^{-1}\frac{b\sin\alpha}{a+b\cos\alpha}$. 172. $\frac{1}{2}$ *mile.*

175. $A=69°\ 10'\ 10''$, $B=46°\ 37'\ 50''$.

176. $\frac{1}{8}\sin\theta+\frac{1}{16}\sin3\theta-\frac{1}{16}\sin5\theta$. 177. 30 *ft.*

178. $3\,\mathrm{L}\sec A-4\,\mathrm{L}\cos B+2\,\mathrm{L}\sin C=4.90309$.

180. 300 *yds;* 4200 *sq. yds.* 182. $\cos\phi+\cos2\alpha=c$.

183. ±1, ∓2. 184. $-\frac{1\pm\sqrt{(1+\tan^2 2A)}}{\tan 2A}$. 185. 1.06.

190. 9.4660179, 17° 8′. 192. $\pm m\sqrt{\frac{n^2+1}{m^2+1}}$.

194. 3 *yds.* 196. 21° 14′ 13″; 10.0305608.

197. $\frac{m+n}{m-n}$ or $\frac{m-n}{m+n}$; $\pm\frac{2mn}{m^2+n^2}$.

199. 45°, or $\{2n+(-1)^n\}\,\frac{1}{4}\pi$. 200. 126° 22′ or 96° 27′.

201. 45°, or 120°, or, generally, $(4n\pm1)\,\frac{1}{4}\pi$ or $(6n\pm2)\,\frac{1}{3}\pi$.

204. 208 *ft.* 205. $(4n+1)\,\frac{1}{8}\pi$ or $(4n-1)\,\frac{1}{8}\pi$.

208. $a\frac{\sin\beta}{\sin(\alpha+\beta)}\tan\theta$, $\phi-\alpha$. 209. 2 *m* 629 *yds.*

210. 36° 7′ 47″. 213. $(2n+1)\,\frac{1}{2}\pi$ or $\frac{6n\pm1}{3(n-1)}\,\pi$.

215. $\sqrt{(5-2\sqrt{5})}$; $\sqrt{\frac{2\sqrt{2}-\sqrt{3}+1}{2\sqrt{2}+\sqrt{3}-1}}$.

216. $u=k\cos(\phi\mp\theta)$, where $\tan\phi=\frac{b}{a}$, $k=a\sec\phi$.

217. 55° 46′ 16″. 221. $k\sin(C+\phi)=\cot a\sin b$, where

222. 0, ∞, $\sqrt{2}$. $\tan\phi=\cos b\tan A$, $k=\cos b\,\mathrm{cosec}\,\phi$.

224. $n\pi+(-1)^n\,\frac{1}{10}\pi$ or $n\pi-(-1)^n\,\frac{3}{10}\pi$. 225. 12492.

232. $\frac{2}{3}\frac{3}{5}\frac{5}{6}$. 233. $a\frac{\sin\phi}{\sin(\theta+\phi)}\left\{\frac{\sin\theta\sin\alpha}{\sin(\theta+\psi)}+\sin\beta\right\}$.

236. $(2n\frac{\pi}{\alpha}-1)\pm\frac{1}{2}\ (\frac{\pi}{\alpha}-1)$. 239. $a=s\sin\frac{1}{2}A\sec\frac{1}{2}B\sec\frac{1}{2}C$.

241. $m\frac{\cos\{\frac{1}{2}(\alpha+\beta)-\gamma\}}{\sin\gamma\cos\frac{1}{2}(\alpha-\beta)}$. 243. $n\pi$ or $(6n\pm1)\frac{1}{3}\pi$.

246. $\sqrt{(a^2+\alpha\beta+\beta^2)}$, $\frac{1}{4}\sqrt{3}(\alpha\beta+\alpha\gamma+\beta\gamma)$.

248. (i) $(4n\pm1)\frac{1}{4}\pi$; (ii) $(6n\pm1)\frac{1}{3}\pi$ or $(6n\pm2)\frac{1}{3}\pi$; (iii) $(8n+1)\frac{1}{4}\pi$.

251. 15°, $\{6n+(-1)^n\}\frac{1}{12}\pi$. 252. $\tan^{-1}\frac{n}{n+2}$.

253. $\tan A=\sqrt{\frac{4n^2-m^2}{4m^2-n^2}}=\cot B$. 256. $C=90°$.

260. $[6n\pm\frac{1}{2}\{3-(-1)^n\}]\frac{1}{3}\pi$. 261. $\frac{3}{8}\sqrt{\{6(p^2+q^2)-3r^2\}}$.

262. $A=n\pi$ or $n\pi+(-1)^n\frac{1}{4}\pi$; $\operatorname{chd}A=0$, or ±2, or $\pm\sqrt{(2\pm\sqrt{2})}$.

263. $u=2\sqrt{a}\cos(45°-\frac{1}{2}\theta)$, where $\cos\theta=\frac{b}{a}$.

265. $\frac{abc\tan a}{4\sqrt{\{s(s-a)\ (s-b)\ (s-c)\}}}$, where $s=\frac{1}{2}(a+b+c)$.

272. $\frac{1}{12}\pi$. 274. 30° or 150°; $(12n\pm1)\frac{1}{6}\pi$ or $(12n\pm5)\frac{1}{6}\pi$.

276. 8 *ft.* 277. 1.062. 278. $\sqrt{3}+2:2$.

279. $\frac{1}{4}\sqrt{\frac{(ab+cd)\ (ac+bd)\ (ad+bc)}{(s-a)\ (s-b)\ (s-c)\ (s-d)}}$. 283. $\sqrt[4]{\frac{q^2}{pq-p^2}}$, $\sqrt[4]{\frac{p^2}{pq-p^2}}$.

284. $\frac{a\sin\alpha\sin\beta}{\sqrt{(\sin^2\alpha+\sin^2\beta-2\sin\alpha\sin\beta\cos\theta)}}=\frac{a\sin\alpha\sin\beta\cos\phi}{2\sin\frac{1}{2}(\alpha-\beta)\cos\frac{1}{2}(\alpha+\beta)}$,

if $\tan^2\phi=\frac{\sin\alpha\sin\beta\sin^2\frac{1}{2}\theta}{\sin^2\frac{1}{2}(\alpha-\beta)\cos^2\frac{1}{2}(\alpha+\beta)}$.

285. $\frac{\sqrt{(abcd)}}{a+c}$ or $\frac{\sqrt{(abcd)}}{b+d}$. 287. 2.4065404.

288. $(4n-1)\frac{1}{4}\pi$. 291. $(4n-1)\frac{1}{4}\pi$ or $(4n-1)\frac{1}{8}\pi$.

296. (i) $a>b$, $u=\frac{\sec^2\frac{1}{2}x\cos^2\phi\sec^2\frac{1}{2}\theta}{2a}$,

where $\cos\theta=\frac{b}{a}$, $\tan\phi=\tan\frac{1}{2}\theta\tan\frac{1}{2}x$:

(ii) $a<b$, $u=\frac{\sec\frac{1}{2}(x+\theta)\sec\frac{1}{2}(x-\theta)}{2b}$, where $\cos\theta=\frac{a}{b}$.

299. $x=0$, ±1, $\pm\sqrt{7}$. 300. $a\frac{\sin\alpha\sin\varepsilon}{\sin(\delta+\varepsilon)}$, $a\frac{\sin\beta\sin\varepsilon}{\sin(\delta+\varepsilon-\gamma)}$.

303. $\pm18°$, $\pm54°$, or, in one general formula, $(n\pm\frac{1}{5}\pm\frac{1}{10})\pi$.

310. 15°, 30°, 45°, 75°, 90°, 105°, 135°, 150°, 165°, 195°, 210°, 215°, 245°, 270°, 275°, 305°, 330°, 335°, 365°, 390°:
$\theta=\frac{1}{2}n\pi$ or $(1\pm\frac{1}{3})(2n+1)\frac{1}{8}\pi$, n being any integer.

313. $152\frac{1}{2}$ *miles.* 314. $\sqrt{(AB)}$, $\frac{2B\sqrt{A}}{\sqrt{A}+\sqrt{B}}$.

317. $\frac{1}{a}+\frac{1}{b}=\frac{1}{m}+\frac{1}{n}$. 319. $B=4°\ 55'\ 11''$, $C=168°\ 27'\ 25''$.

320. 30°, 45°, 60°, or $(n+\frac{1}{6})\pi$, $(n+\frac{1}{4})\pi$, $(n+\frac{1}{3})\pi$.

322. 5 *m* 1156 *yds.* 324. $\pm\sqrt{(ab)}$.

329. $\theta=(m+n)\frac{1}{2}\pi$, $\phi=(n-m)\frac{1}{2}\pi$; 330. $x=\pm 1$ or $\frac{1}{2}$.
or $\theta=\frac{1}{2}(m+n)\pi+\{2+(-1)^m\}\frac{1}{6}\pi$, $\phi=\frac{1}{2}(m-n)\pi+\{2-(-1)^m\}\frac{1}{6}\pi$.

331. $\frac{m\sin A}{n-m\cos A}=\frac{\sin A\sin\phi}{\sin(A-\phi)}$, if $\tan\phi=\frac{m}{n}\sin A$.

333. 6.34 *miles.* 338. $h'=\sqrt{\frac{(b-c)\ (2bc-ab-ac)}{a}}$, $h=\frac{a}{b}h'$.

340. $\frac{1}{6}(4\pi-3\sqrt{3})a^2$. 341. 4 *sq. in.*, $1\frac{3}{11}$ *sq. in.*

343. 15° or $(12n\pm 1)\frac{1}{12}\pi$. 346. $x=\frac{1}{4}(3\pm\sqrt{17})$, $y=-\frac{1}{4}(3\pm\sqrt{17})$.

348. 1 *m* 1527 *yds.* 350. ± 1, $\pm 2\sin^2\frac{1}{2}a$, $\pm 2\cos^2\frac{1}{2}a$.

354. $\pm(\sqrt{3}+1)$. 357. $\tan\theta=\pm 2$ or ± 1, $\tan\phi=\pm 1$ or $\pm\frac{1}{2}$.

360. $r^2(3+\sqrt{6})$.

361. $\sin(\theta+\phi)=\frac{\sin(\alpha+\beta)}{\sqrt{2}\cos\alpha\cos\beta}$, where $\cos 2\phi=\tan^2\alpha\tan^2\beta$.

366. $(a^2-cd)^2=b^2\{(a+c)^2+(a+d)^2\}$. 370. 9.4 *m*, 15.2 *m.*

372. $0, \infty, \pm 1, \pm(\sqrt{2}\pm 1)$. 373. $\frac{m^2}{a^2}+\frac{1}{a^4}$. 374. $\frac{2ab\sin\alpha}{a^2-b^2}$.

378. $+\tan\frac{1}{4}A, -\cot\frac{1}{4}A, \pm\tan\frac{1}{4}(\pi\mp A)$. 379. $a=\pm\frac{1}{4}\pi, a'=-\frac{1}{6}\pi$ or $\sin^{-1}(\frac{1}{6})$

381. $\sqrt{\frac{ab\ (a+b)}{a\cot^2\gamma+b\cot^2\alpha-(a+b)\cot^2\beta}}$. 382. $a\{1\pm\sqrt{\frac{2b}{1+b}}\}$.

386. $\frac{(av-bu)\sin\theta}{\sqrt{(u^2+v^2-2uv\cos\theta)}}$. 388. $\{6n+(-1)^n\}\frac{1}{12}\pi$. 389. $C=90°$.

397. $\sqrt{\frac{2}{a}}$. 398. $a\tan\alpha\sqrt{(2-\sqrt{2})}$. 399. $\tan\theta\ \mathrm{cosec}\frac{\theta}{2^n}$.

402. $(2n+1)\pi-\alpha$ or $\sin^{-1}\{-\sin\alpha\cos\alpha\}-\alpha$. 404. 232 *ft.*

406. $\frac{4}{7}\frac{3}{0}\frac{2}{0}$ *sq. yds.* 412. $\theta=(2n+1)\frac{1}{2}\pi$. 423. $\tan^{-1}n\alpha$. 424. $2^{\frac{1}{2}n}\cos\frac{1}{4}n\pi$.

425. $\frac{1}{4}\{2n+1-\sin(2n+1)\theta\ \mathrm{cosec}\theta\}$. 427. $2\cos 40°, -2\cos 20°, 2\sin 10°$.

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439. $\frac{3\sin\frac{1}{2}(n+1)\ \theta\sin\frac{1}{2}n\theta}{4\sin\frac{1}{2}\theta}-\frac{\sin\frac{3}{2}(n+1)\ \theta\sin\frac{3}{2}n\theta}{4\sin\frac{3}{2}\theta}$.

441. $\frac{\sin\frac{1}{2}na}{\sin\frac{1}{2}a}\{\cos\frac{1}{2}(n+1)\alpha+\sqrt{-1}\sin\frac{1}{2}(n+1)\alpha\}$. 442. $\frac{\sin 2nA}{2\sin A}$.

444. $\frac{1}{2}\ (xe^{x\cos\theta}+x^{-1}e^{x^{-1}\cos\theta})$. 448. $\frac{\sin\ (n+1)\ \theta\sin n\theta}{\sin\theta\cos\theta\cos\ (2n+1)\ \theta}$.

451. $4\log_e\frac{\sqrt{a}+\sqrt{b}}{2\sqrt[4]{a}}+4(m\cos^2\theta-\frac{1}{2}m^2\cos^2 2\theta+\&c)$, where $m=\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$

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